MATH 3240 March 17

§7.3

<u>Proposition</u>: Every continuous mapping from a compact space into a Hausdorff space is closed.

<u>Solution</u>: Suppose $f: X \longrightarrow Y$ is continuous, X compact and Y Hausdorff. Let $F \subset X$ be a closed subset, hence compact.
Then $f(F)$ is compact,
and since Y is Hausdorff $f(F)$ is closed.

<u>Proposition</u>: If $f: X \longrightarrow Y$ is a continuous mapping from a compact space X <u>onto</u> a Hausdorff space Y, then $X/_{\sim_f}$ is homeomorphic to Y. (Recall: $\sim_f$ is $x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2)$).

<u>Proof</u>: By a previous proposition, the map $g: X/_{\sim_f} \longrightarrow Y$ defined by $g([x]) = f(x)$ is well-defined, one to one and continuous. Moreover it is open/closed exactly when $f: X \longrightarrow Y$ is open/closed.

By the previous proposition, f is closed since X is compact and Y is Hausdorff. Thus g is closed, hence it is a homeomorphism.

<u>Example</u>: Consider

$$f: S^2 \longrightarrow \mathbb{R}^6 \text{ where } f(x,y,z) = (x^2, y^2, z^2, xy, xz, yz).$$

Then f is continuous since the component functions are continuous. Since S^2 is a closed, bounded subset of $\mathbb{R}^3$ it is compact, and since Hausdorffness is hereditary $f(S^2) \subseteq \mathbb{R}^6$ is Hausdorff. Therefore by the previous proposition, $f(S^2) \cong S^2/\sim_f$.

What is the equivalence relation $\sim_f$?

Suppose $f(x,y,z) = f(x',y',z')$. Then

$$x^2 = (x')^2, \quad y^2 = (y')^2, \quad z^2 = (z')^2.$$

Thus $x = \pm x'$, $y = \pm y'$, $z = \pm z'$. Then the equations

$$xy = x'y'$$
$$xz = x'z'$$

and $yz = y'z'$

force $(x,y,z) = (x',y',z')$ or $(-x,-y,-z) = (x',y',z')$.

Thus $\sim_f$ is the equivalence relation of identifying antipodal points.

Therefore $f(S^2) \cong S^2/\sim_f$ is an embedding of the projective plane $\mathbb{RP}^2$ into $\mathbb{R}^6$.

In fact we can even define

$$g : S^2 \longrightarrow \mathbb{R}^4$$
$$g(x,y,z) = (xy, xz, y^2 - z^2, 2yz)$$

which satisfies $g(x,y,z) = g(-x,-y,-z)$

and so gives an embedding $\mathbb{R}P^2 \cong g(S^2) \subseteq \mathbb{R}^4$.

<u>Question</u>: Can $\mathbb{R}P^2$ be embedded in $\mathbb{R}^3$? (Ans: no!).

In general, the question of when a manifold (i.e. a space locally homeomorphic to $\mathbb{R}^n$) can be embedded in $\mathbb{R}^m$ is extremely hard, but there are some big famous theorems:

<u>Theorem</u> (Whitney embedding theorem)
Any (smooth) n-manifold is homeomorphic to a subset of $\mathbb{R}^{2n}$!

So, since we know $\mathbb{R}P^2$ is a 2-manifold it <u>should</u> be embeddable in $\mathbb{R}^4$, as we just saw.

<u>Example</u>: (Better proof of assignment question).
One can set up the proof that $X_f \cong S^3$ (Question 1.1, asst 3) using this approach. The required maps are included in the solutions from March 10.

<u>Def</u>: A space X is <u>Lindelöf</u> if every open cover has a countable subcover.

A space is <u>countably</u> <u>compact</u> if every countabee open cover has a finite subcover.

Example: Let $X = \mathbb{R}$ with basis $\{[a,b) \mid a<b\}$, ie. the Sorgenfrey line.

Claim 1: X is not compact.

Proof: The sets $\mathcal{W} = \{[x, x+1) \mid x \in \mathbb{Z}\}$ provide an open cover with no finite subcover.

Claim 2: X is Lindelöf.

Proof: Let $\mathcal{W}$ be an open cover of X, and set

$$\mathcal{V} = \{(a,b) \subseteq \mathbb{R} \mid (a,b) \subset U \text{ for some } U \in \mathcal{W}\}$$

Let $X^* = \bigcup_{(a,b)\in\mathcal{V}} (a,b) \subset X$.

For all $x \in X^*$, choose an interval (a,b) with $x \in (a,b)$ and choose an open interval I_x with rational endpoints such that $x \in I_x \subset (a,b)$. The collection $\{I_x\}_{x\in X^*}$ is countable and covers X^*.

For each I_x, choose one of $(a,b) \in \mathcal{V}$ with $I_x \subset (a,b)$; call this new collection $\mathcal{V}'$. Then $\mathcal{V}'$ is a countable subcover of $\mathcal{V}$. Now, for each element of $\mathcal{V}' \subset \mathcal{V}$, choose an element of $\mathcal{W}$ containing it; call this new collection $\mathcal{W}'$. Then $\mathcal{W}'$ is a countable subset of $\mathcal{W}$ covering X^*.

Now we need only cover $X \setminus X^*$ with a countable subset of $\mathcal{W}$ to complete the proof:

For every $x \in X \setminus X^*$ and $U \in \mathcal{W}$, consider the interval $[x, b_u) \subset U$ ($b_u > x$ chosen so that $[x, b_u) \subset U$).

Given $x, y \in X \setminus X^*$, if $x < y$ then $[x, b_u) \cap [y, b_v) = \phi$ for all $V \in \mathcal{W}$. For if not, then $y \in [x, b_u)$ and $y \in (x, b_u) \subset U$
$\Rightarrow y \in X \setminus X^*$, contradiction.

So, the intervals $[x, b_u)$ and $[y, b_v)$ are pairwise disjoint ~~wh~~ whenever $x \neq y$, choose one such interval for each $x \in X \setminus X^*$.
Any collection of pairwise disjoint intervals in $\mathbb{R}$ is countable (each contains some $q \in \mathbb{Q}$ for example), so $X \setminus X^*$ is countable. So it suffices to choose, for each $x \in X \setminus X^*$, a set $W \in \mathcal{W}$ that covers it.

$\blacksquare$

MATH 3240 Topology 1.

Last day we ended with a generalization of compactness:

A space X is <u>Lindelöf</u> if every open cover has a countable subcover.

<u>Example</u>: The Sorgenfrey line is not compact, but is Lindelöf. Note it is <u>not</u> second countable, because...

<u>Proposition</u>: (Lindelöf lemma) Every second countable space is Lindelöf.

<u>Proof</u>: Let X be a second countable space and $\mathcal{W}$ an open cover of X. Let $\mathcal{B}$ be a countable basis of X. Each $W \in \mathcal{W}$ is a union of elements of $\mathcal{B}$, so create a new countable cover $\mathcal{V}$ of X that consists of all $B \in \mathcal{B}$ that are used in writing some $W \in \mathcal{W}$ as a union of basic elements. I.e, if

$$W = \bigcup_{i \in I} B_i \text{ , then } \{B_i\}_{i \in I} \subset \mathcal{V}.$$

Now choose a countable subcover $\mathcal{W}' \subset \mathcal{W}$ as follows: for each $V \in \mathcal{V}$, choose $W \in \mathcal{W}$ s.t. $V \subset W$. Then the collection of all such W's is $\mathcal{W}'$, which is countable.

<u>Example</u>: This shows that $\mathbb{R}$ is Lindelöf, but is not compact.

Recall the Bolzano Weierstrass theorem:

Every bounded sequence in R^n has a convergent subsequence.

Inspired by this, we define a Bolzano-Weierstrass space (or BW space), to be a space X in which every infinite subset has an accumulation point.

First, we note:

Def: x is an accumulation point of A if every open subset containing x contains a point of A other than x.

Lemma: Suppose X is Hausdorff and $A \subseteq X$. If $p \in X$ is an accumulation point of A, then every neighbourhood of p contains infinitely many points of A.

Proof: Construct infinitely many points $\{x_n\}$ as follows, given a nbhd U of p:

$\exists\, x_1 \in U \cap A$ since p is an accumulation point. Now suppose we have $x_1, \dots, x_k$. For each x_i, $\exists$ nbhds U_i of x_i and V_i of p such that $U_i \cap V_i = \emptyset$. Set

$$V = \left(\bigcap_{i=1}^{k} V_i\right) \cap U,$$ which is an open neighbourhood of p and so contains a point x_{k+1} of A and not any of $x_1, \dots, x_k$. //

<u>Proposition</u>: Suppose that X is a Hausdorff space. Then X is a BW-space iff it is countably compact.

<u>Proof</u>: ($\Rightarrow$) Suppose X is a BW-space. Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be a countable open covering of X, assume that no U_i is contained in $U_1 \cup U_2 \cup \ldots \cup U_{i-1}$ (eliminate redundancy). Suppose $\mathcal{U}$ has no finite subcovering.

Then there's a set

$$A = \{x_n \in X \mid x_n \in U_n \setminus (\bigcup_{i=1}^{n-1} U_i)\},$$

and the set A is infinite since $x_i \neq x_j$ if $i \neq j$. Since X is a BW space, A has an accumulation point x. Since $\mathcal{U}$ is a cover, there exists n such that $x \in U_n$, and therefore U_n contains infinitely many points of A (here use the lemma). However for $m > n$, $x_m \notin U_n$ by construction, a contradiction.

($\Leftarrow$) Suppose that X is countably compact. We show that every countably infinite subset has an accumulation point. Suppose $A = \{a_1, a_2, \ldots\}$ does not have an accumulation point. Then A is closed since $\bar{A} = A \cup A' = A$, and since each $a_i \in A$ is not an accumulation point of A $\exists$ an open nbhd U_i of a_i s.t. $U_i \cap A = \{a_i\}$. Thus $\{U_i\}_{i=1}^{\infty} \cup \{(X \setminus A)\}$ is an open covering of X, so it must have a finite subcovering, say $\{U_1, \ldots, U_n\} \cup \{X \setminus A\}$.

But then some U_i must contain infinitely many of the a_i's, a contradiction.

<u>Proposition</u>: Every compact space is a BW space.

<u>Proof</u>: Suppose X is not a BW space. Then there is an infinite subset of A without accumulation points in X. Thus, A contains all its accumulation points and so is closed.

Now if X is compact, then A is compact since it is closed. Moreover since A has no accumulation points, for each $a \in A$ $\exists\, U_a$ an open nbhd of a such that $U_a \cap A = \{a\}$. Then $\{U_a\}_{a \in A}$ is an open cover of A, so we choose a finite subcover $\{U_{a_1}, \dots, U_{a_n}\}$. Then

$$A = \left(\bigcup_{i=1}^{n} U_{a_i}\right) \cap A = \bigcup_{i=1}^{n} (A \cap U_{a_i}) = \{a_1, \dots, a_n\},$$ so that

A is finite, a contradiction.

Our goal now is to show that for metric spaces, the converse also holds: If (X,d) is a Bolzano-W. space, then X is compact. This requires a famous lemma

<u>Lemma</u> ~~(Lesbegu~~ (Lebesgue number lemma)

~~For every~~ Let (X,d) be a BW-metric space, and suppose that $\mathcal{W}$ is an open cover of X.

Then $\exists\, \varepsilon > 0$ s.t. $\forall x \in X\ \exists\, W \in \mathcal{W}$ with $B(x, \varepsilon) \subset W$.

(i.e. There's a radius $\varepsilon > 0$ called the Lebesgue number such that every ε-ball is contained in some element of the cover).

Proof: Let X be a BW-space with metric d, $\mathcal{W}$ an open cover, and assume that $\exists\, x \in X$ s.t. $\forall \varepsilon > 0$ the ball $B(x, \varepsilon)$ is not a subset of any $U \in \mathcal{W}$. In particular, for every $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}^+$, there is a point $x_n \in X$ such that $B(x_n, \frac{1}{n})$ is not contained in any $U \in \mathcal{W}$.

First, note that $\{x_i\}_{i=1}^{\infty}$ is an infinite set. If not, then $x_m = x_n$ $\forall m \geq n$ for some n, and the statement "$B(\overset{x_n}{\overset{\shortparallel}{x_m}}, \frac{1}{m})$ is not a member of any $U \in \mathcal{W}$ $\forall m \geq n$" contradicts the fact that the balls $\{B(x_n, \frac{1}{m})\}_{m \in \mathbb{N}^+}$ form a local basis.

So, since X is a BW-space the sequence $\{x_i\}_{i=1}^{\infty}$ has an accumulation point, say $x \in X$. Choose $U \in \mathcal{W}$ containing x, and a ball $B(x, r) \subset U$. Since X is ~~Choose~~ a metric space, $B(x, \frac{r}{2})$ contains infinitely many of the points $\{x_i\}_{i=1}^{\infty}$.

Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$, such that $x_m \in B(x, r/2)$.

Then $B(x_m, \frac{1}{m}) \subset B(x, r) \subset U \in \mathcal{W}$, contradicting our choice of x_m.