Math 3390

Introduction to topology, final exam study questions

Question: Let $f : X \to Y$ be a homeomorphism and $A \subset X$. Show that $f(\partial A) = \partial f(A)$.
Solution: Let $x \in \partial A$. Fix an arbitrary open neighbourhood V of $f(x)$, and left $U = f^{-1}(V)$. Then U is open by continuity of f, and $x \in U$. Since $x \in \partial A$, $U \cap A$ contains some $a \neq x$, and $U \cap A^c$ contains some $b \neq x$. Then $f(a) \in V \cap f(A)$, and $f(b) \in V \cap f(A^c) = V \cap f(A)^c$, where $f(a) \neq f(x)$ and $f(b) \neq f(x)$. Hence $f(x) \in \partial f(A)$, and so $f(\partial A) \subset \partial f(A)$.

For the reverse inclusion, let $y \in \partial f(A)$ and choose $x = f^{-1}(y)$. Fix an arbitrary neighbourhood U of x and let $V = f(U)$, which is open. Note that $y \in V$. Since $y \in \partial f(A)$, $V \cap f(A)$ contains some $a \neq y$ and $V \cap f(A)^c$ contains some $b \neq y$. Then $f^{-1}(a) \in U \cap A$, and $f^{-1}(b) \in U \cap f^{-1}(f(A)^c) = A^c$, and $f^{-1}(a) \neq x$ and $f^{-1}(b) \neq x$. Therefore $x \in \partial A$, so $y \in f(\partial A)$. Hence $\partial f(A) \subset f(\partial A)$.

Question:

(a) Let (X, τ) be a topological space. Let $\mathcal{X}$ be the collection of closed sets in X. Let C be the collection of closed sets satisfying: Every $V \in \mathcal{X}$ is an intersection of elements of C. Show that $\mathcal{B} = \{A^c \mid A \in C\}$ is a basis for τ.

Solution: Fix $U \in \tau$ and let $V = U^c$, which is closed. Then $V = \bigcap_{i \in I} A_i$ for some sets $A_i \in C$. By De Morgan's laws, this gives $U = V^c = \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$, where now $A_i^c \in \mathcal{B}$. Thus U is a union of elements of $\mathcal{B}$, so $\mathcal{B}$ is a basis for τ.

(b) Let C be a collection as in part (a). Show that for all $V \in \mathcal{X}$, and for all $x \notin V$, there exists $A \in C$ such that $V \subset A$ and $x \notin A$.

Solution: Fix $V \in \mathcal{X}$, and $x \notin V$. Let $U = V^c$, so U is open and $x \in U$. Since $\mathcal{B} = \{A^c \mid A \in C\}$ is a basis, there exists $A^c \in \mathcal{B}$ such that $x \in A^c \subset U$. Then $U^c \subset (A^c)^c$, that is, $V \subset A$, and $x \notin A$.

Question: Recall that an equivalence relation $\sim$ on a set X defines (or is defined by) a subset $R \subset X \times X$, with $x \sim y$ if and only if $(x, y) \in R$. Bearing this in mind, let X be a T_3 space and let $p : X \to X/\sim$ be a closed quotient map. Prove that the corresponding subset $R \subset X \times X$ is closed.
Solution: We show the complement is open, so let $(x, y) \notin R$. We will find a basic open neighbourhood $U \times W$ of (x, y), with $U \times W \subset R^c$. Equivalently, we need open sets U and W containing x and y respectively such that no point of X is equivalent to a point of Y. This means $p(U) \cap p(W) = \emptyset$.

Now since $(x, y) \notin R$, we have that $p(x) \neq p(y)$, in other words $x \notin p^{-1}(p(y))$. Since X is regular the set $\{y\}$ is closed, and so is $\{p(y)\}$ since p is a closed map, and thus $p^{-1}(p(y))$ is a closed set. Again by regularity, there exist open sets U and V with $x \in U$ and $p^{-1}(p(y)) \subset V$.

Now we appeal to the following property of closed maps: Suppose $p : X \to Y$ is a closed map. Given any $S \subset Y$ and any open U containing $p^{-1}(S)$, there exists an open set V containing S such that $p^{-1}(V) \subset U$ (the proof of this is one line).

In our situation, this property of closed maps gives an open neighbourhood W of $p(y)$ such that $p^{-1}p(y) \subset p^{-1}(W) \subset V$. Then the open neighbourhood $U \times p^{-1}(W)$ gives the neighbourhood of (x, y) that we needed.

Question: Let (X, τ) be a topological space. Suppose that (X, τ) is normal. Show that for every closed subset $F \subset X$ and every open set U with $F \subset U$, there is an open set W with $F \subset W \subset \overline{W} \subset U$.
Solution: Let F closed and U open be given, satisfying $F \subset U$. Then F and U^c are disjoint closed sets. By normality of X, there exist open sets V and W such that $F \subset W$ and $U^c \subset V$. Since $W \cap V = \emptyset$, $W \subset V^c$. But V^c is closed, so $\overline{W} \subset V^c$, and $V^c \subset U$ since $U^c \subset V$. Therefore

$$F \subset W \subset \overline{W} \subset V^c \subset U,$$

as was needed,

Question: Prove the Lebesgue number lemma. There are many proofs, here is one:
`https://proofwiki.org/wiki/Lebesgue's_Number_Lemma`

Question: Prove that a metric space is compact if and only if it is countably compact (you are allowed to use high-powered theorems here).
Solution: Let X be a metric space. Suppose that X is compact. Then every countable cover has a finite subcover, since every open cover has a finite subcover, so X is countable compact.

On the other hand, suppose that X is countably compact. Since X is a metric space it is Hausdorff. Thus X is a BW space, because for Hausdorff spaces, the BW property is equivalent to countable compactness. But now a metric space with the Bolzano-Weierstrass property is compact.

Question: Let X and Y be topological spaces. Suppose that $A \subset X$ is closed. If $f : A \to Y$ is a continuous map, show that the composition of maps given by

$$Y \overset{inclusion}{\longrightarrow} X \oplus Y \overset{quotient}{\longrightarrow} X \cup_f Y$$

is an embedding.
Solution: We write this composition of maps as $h : Y \to X \cup_f Y$, where the formula is $h(y) = [y]$. First, we show that h is injective, so let $y_0, y_1 \in Y$ be given and suppose that $h(y_1) = h(y_0)$. Then

$$[y_1] = \{y_1\} \cup f^{-1}(y_1) = \{y_0\} \cup f^{-1}(y_0) = [y_0]$$

but since each of $f^{-1}(y_1)$ and $f^{-1}(y_0)$ are subsets of X, in order to have equality we must have that $\{y_0\} = \{y_1\}$, so $y_1 = y_0$. For the map h to be an embedding, we also need it to be continous, but it is a composition of continuous maps so this is immediate.

The last thing to check is that h is a homeomorphism onto its image. We will do this by showing that h is a closed map. So, let $V \subset Y$ Be a closed set. Then

$$h(V) = \{[y] \in X \cup_f Y \mid y \in V\}.$$

By definition of the quotient topology, this is closed in $X \cup_f Y$ if and only if $\bigcup_{[y] \in h(V)} [y] = \bigcup_{y \in V} (\{y\} \cup f^{-1}(y))$ is a closed set in $X \oplus Y$. But we can rewrite:

$$\bigcup_{y \in V} (\{y\} \cup f^{-1}(y)) = \bigcup_{y \in V} \{y\} \cup \bigcup_{y \in V} f^{-1}(y) = V \cup f^{-1}(V)$$

But since f is a continuous map, the set $f^{-1}(V)$ is closed in A. Thus there is a set F closed in X with $F \cap A = f^{-1}(V)$. Since A is closed, this shows that $f^{-1}(V)$ is closed in X. Therefore the intersection

$(V \cup f^{-1}(V)) \cap X = f^{-1}(V)$ yields a closed set in X, and the intersection $(V \cup f^{-1}(V)) \cap Y = V$ yields a closed set in Y. Therefore $V \cup f^{-1}(V)$ is closed in $X \oplus Y$, so $h(V)$ is closed in $X \cup_f Y$.

Question: State the "invariance of domain" theorem, and apply it to show that $\mathbb{R}^n \cong \mathbb{R}^n$ if and only if $n = m$.
Solution: The invariance of domain theorem says:

Theorem 1 *If $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^n$ is an embedding, then $f(U)$ is open in $\mathbb{R}^n$.*

Here is how you apply it as asked. Without loss of generality, suppose that $n < m$ and that there exists a homeomorphism $g : \mathbb{R}^n \to \mathbb{R}^m$. Let $U \subset \mathbb{R}^m$ be a nonempty open set. Then $g^{-1}(U)$ must be a nonempty open set in $\mathbb{R}^n$. Now, fix $k_{n+1}, k_{n+1}, \ldots, k_m \in \mathbb{R}$ and define a map $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, k_{n+1}, k_{n+1}, \ldots, k_m)$. Then f is clearly an embedding, and g is assumed to be a homeomorphism, so the composition $f \circ g^{-1} : \mathbb{R}^m \to \mathbb{R}^m$is an embedding. By the invariance of domain theorem, this implies that $f \circ g^{-1}(U)$ is open in $\mathbb{R}^m$. This is not possible, since the image of f is $\mathbb{R}^n \times \{k_{n+1}\} \times \ldots \times \{k_m\}$, which has empty interior.

Question: Show that a finite union of compact subspaces of a space X is compact.
Solution: Let $X_1, \ldots, X_n$ be compact subsets of a space X. Suppose that $\{U_j\}_{j \in J}$ is an open covering of $\bigcup_{i=1}^n X_i$. Then $\{U_j\}_{j \in J}$ is an open covering for each X_k as well. So, because each X_k is compact, there is a finite set $J_k \subset J$ such that $\{U_j\}_{j \in J_k}$ covers X_k. Set $I = \bigcup_{k=1}^n J_k$. Then $\{U_i\}_{i \in I}$ is a finite collection, and it covers $\bigcup_{i=1}^n X_i$ by construction. Thus the union is compact.

Question: Show that a second countable space is Lindelöf.
Solution: Let $\{B_n\}$ be a countable basis and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering for a space X. For each basic element B_n, if B_n is contained in some set U_i then choose a set $V_n \in \mathcal{U}$ containing B_n, if there is no such set containing B_n then choose nothing.

Observe that the family $\{V_n\}$ covers X: Suppose there is $x \in X$ that is not contained in V_n for some n. Choose U_i containing x, and observe that since $\{B_n\}$ form a basis and U_i is open, there exists B_k such that $x \in B_k \subset U_i$. Then then x would be contained in some V_k by our construction, a contradiction.

Thus the family $\{V_n\}$ is the required countable subcover.

Question: Show that a topological space X is disconnected if and only if X contains a nonempty proper clopen subset.
Solution: ($\Rightarrow$) Let X be a disconnected topological space, and let $\{A, B\}$ be a separation of X. Then by definition, A and B are nonempty and open, $A \cap B = \emptyset$ and $A \cup B = X$. Thus $A^c = B$, so that B is also closed, moreover B is proper since $A \neq \emptyset$.

($\Leftarrow$) Suppose there exists a nonempty proper clopen subset A of X Let $B = A^c$. Then $\{A, B\}$ is a separation of X. Since A is nonempty and proper, $A^c = B$ is also nonempty and proper. Moreover $A \cap B = \emptyset$ and $A \cup B = X$, by construction. Thus $\{A, B\}$ is a separation of X.

Question: Let $X_1, \ldots, X_n$ be a finite collection of first countable spaces. Show that the product $\prod_{i=1}^n X_i$ is a first countable space as well.
Solution: Let $\mathbf{x} = (x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ be given. Since each X_i is first countable, there are

countable local bases $\mathcal{B}_i$ of $x_i \in X_i$ for all $i = 1, \ldots, n$. Consider the collection of products:

$$\mathcal{B}_x = \{\prod_{i=1}^{n} B_i \mid B_i \in \mathcal{B}_i\},$$

we will show this is a local basis at $\mathbf{x}$.

First note that $\mathcal{B}_x$ is countable, since it contains all finite products where the factors range over countable sets. Next, let U be an open neighbourhood of $\mathbf{x}$. Then U contains a basic open neighbourhood $\prod_{i=1}^{n} U_i$ containing $\mathbf{x}$, where U_i is open in X_i for all i. For each set U_i, which contains x_i, there exists $B_i \in \mathcal{B}_i$ with $x_i \in B_i \subset U_i$. Thus we have

$$\mathbf{x} \in \prod_{i \in I} B_i \subset \prod_{i=1}^{n} U_i \subset U$$

where $\prod_{i \in I} B_i$ is an element of $\mathcal{B}_x$. Thus $\mathcal{B}_x$ is a countable local basis, and so $\prod_{i=1}^{n} X_i$ is first countable.