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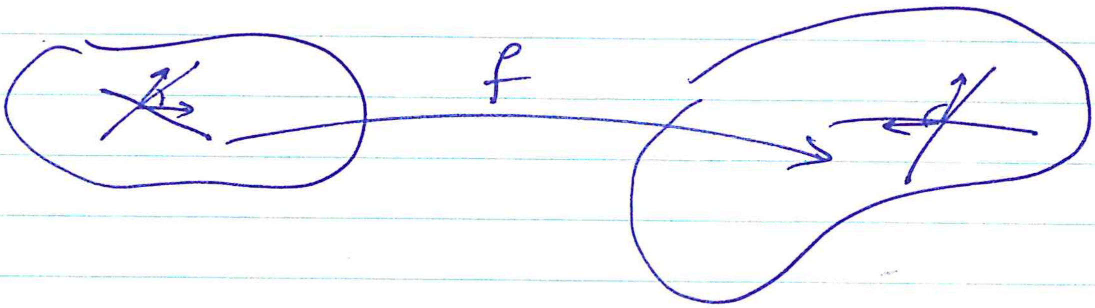
Motivation:

Complex analytic maps - "holomorphic".



Recall this means that f preserves angles, this follows from the Cauchy-Riemann equations.

In fact, we can take alesson from this:

Lesson: complex analytic maps require a notion of angle in order ~~the d~~ to define them.

So, what's an angle?

Angles in a 2d vectorspace. Let $(V, \langle , \rangle)$ be an inner product space, define

define: $\theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \|y\|}\right)$

From here forward, we'll use "g" for the inner product.

Exercise: Two inner products g_1 and g_2 give the same angle for every $X, Y \in V \iff g_1 = \lambda g_2$ for some $\lambda > 0$.

So, "angle on V" is an equivalence class of inner products on V:

$$g_1 \sim g_2 \iff g_1 = \lambda g_2 \text{ for some } \lambda > 0.$$

This is usually called "conformally equivalent," but we reserve this terminology for a later concept.

In fact, rotation by $90°$ is enough to specify all angles:

Given linear $J : V \longrightarrow V$, $J^2 = -I$ (so it's a candidate for rotation by $90°$). Then $\forall X \in V \setminus \{0\}$, the set

$$\{X, JX\} \text{ is a basis. (exercise).}$$

Define an inner product g by $g(JX, X) = 0$

$$g(X, JX) = 0, \quad g(X, X) = 1, \quad g(JX, JX) = 1,$$

and just extend linearly. This gives an inner product relative to which J is $90°$ rotation!

<u>Remark</u>: In fact, $e^{\theta J}$ is rotation by θ for $\theta \in \mathbb{R}$.

<u>Def</u>: • A <u>surface</u> is a 2-dimensional connected smooth manifold.

• An almost complex structure on a surface R is: a smoothly varying linear map

$$J_p : T_p R \longrightarrow T_p R \quad \forall p \in R$$

s.t. $J_p^2 = -I_p \quad \forall p \in R$.

- A Riemannian metric on R is a smoothly varying $g_p : T_pR \times T_pR \longrightarrow \mathbb{R}$ which is an inner product.
- Two metrics g, h (ie. inner products at each point) are <u>conformally equivalent</u> if $\exists\ \lambda : R \longrightarrow \mathbb{R}$, $\lambda > 0$ s.t.

$$g = \lambda h.$$

- A conformal structure on R is an equivalence class of Riemannian metrics on R. + orientation.

<u>Theorem</u>: A conformal structure uniquely determines an almost structure, and an almost complex structure uniquely determines a conformal structure.

(The "determination" is from doing what we did before on V, using J to define g, but we do it on each tangent space T_pR)

<u>Proof sketch</u>: Pick p and fix X_p. Starting from a conformal structure, we want to get angles.

nope ~~Define J_pX_p to be the unique vector, up to scale, such that $\{X_p, J_pX_p\}$ is positively~~ oriented; and $g_p(X_p, J_pX_p) = 0$.

Fix a g in the equivalence class and an orthonormal basis $\{X_p, Y_p\}$, and assume it's positively oriented.

Define $J_pX_p = Y_p$, $J_pY_p = -X_p$

and extend complex linearly.

Conversely, given J, fix a smooth non-vanishing vector field X on R, and define g_p using X_p, J_pX_p to be orthonormal on each tangent space.

Examples: ① R = unit sphere in $\mathbb{R}^3$, centered at $(0,0,0)$. Under spherical coordinates, an open set $V \subset \mathbb{R}^2$

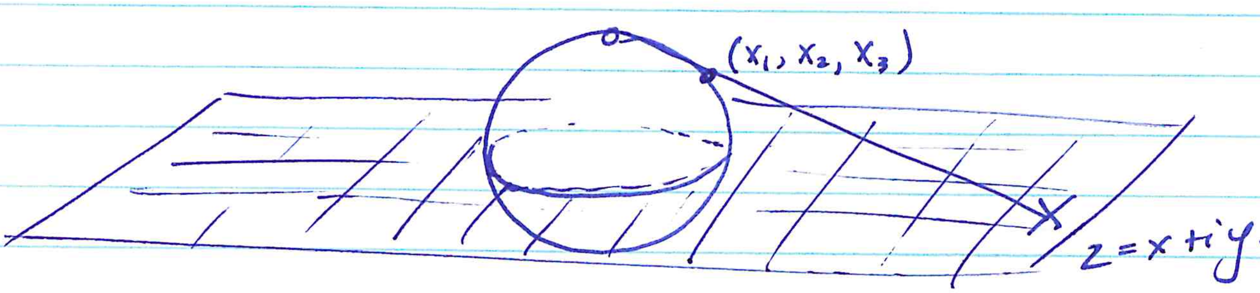
$$V=(0,\pi)\times(0,2\pi)$$

can be identified with part of R.

The Euclidean inner product on R has the form $g=\sin^2\phi\, d\theta^2+d\phi^2$

$$(\phi,\theta)\longmapsto(\cos\theta\overset{\sin}{\cancel{\cos}}\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

② R = unitsphere again, and let $V=\mathbb{C}$, the Image under stereographic projection



$$g_s=2\,\frac{dx^2+dy^2}{(1+x^2+y^2)^2}=2\,\frac{|dz|^2}{(1+|z|^2)^2}$$

③ R = disk

$$= \{(x,y) \mid x^2+y^2 \leq 1\}$$

Then

$$g_h = 2\,\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$$

$$= \frac{2\,|dz|^2}{(1-|z|^2)^2}$$

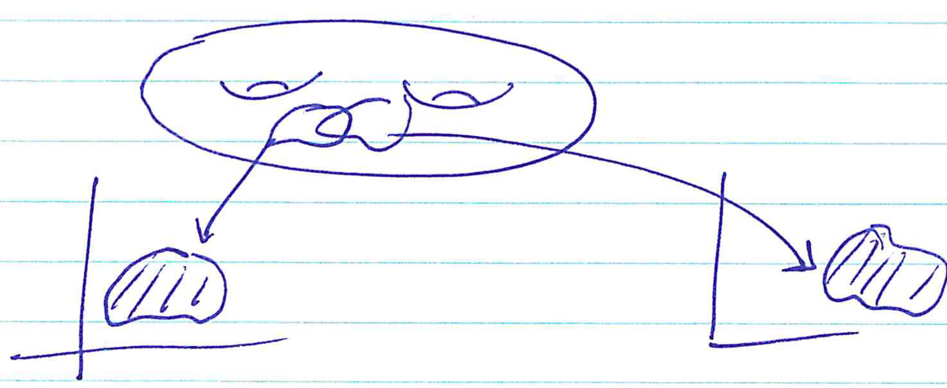
④ $\mathbb{C} = \mathbb{R}^2$ with Euclidean metric $g_e = dx^2+dy^2 = |dz|^2$.

Exercise: Given $g_k = \lambda_k(dx^2+dy^2)$, if a diffeo $f: V_1 \longrightarrow V_2$ satisfies $f^* g_2 = g_1$, then f is holomorphic.

Definition: Let R be a surface, with Riemannian metric g. We say coordinates $\phi: U \longrightarrow V \subseteq \mathbb{R}^2$ are isothermal if $g = \lambda(dx^2+dy^2)$ in these coordinates.

Definition: A Riemann surface R is a surface with an atlas of charts $\{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ where

- $\phi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \cancel{\mathbb{R}^2}\ \mathbb{C}$ is a homeomorphism
- U_α's cover R
- $\phi_\beta \circ \phi_\alpha^{-1}$ is a biholomorphism.



A Riemann surface defines an almost complex structure.

<u>Conversely</u>: Given a J, $\exists$ a Riemann surface that induces J.

<u>Sketch</u>: J determines a conformal structure. Choose a g in the equivalence class. Locally

$$g = E\,dx^2 + F\,dx\,dy + G\,dy^2$$

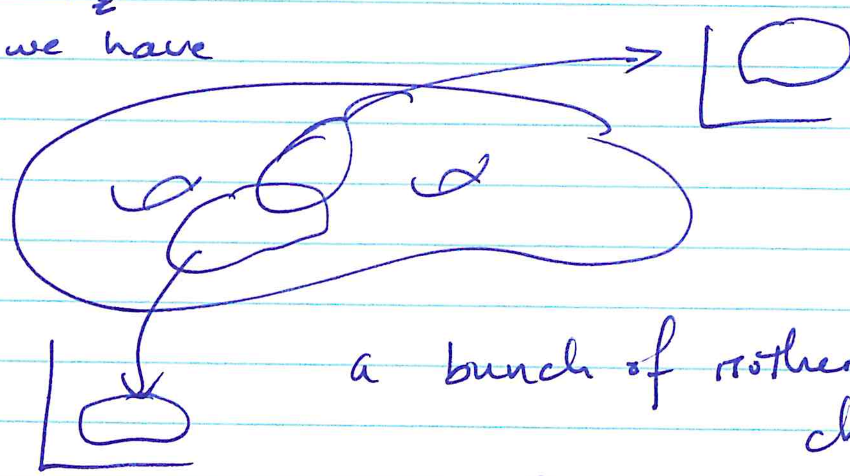
$$= \rho^2 |dz + \mu\, d\bar{z}|^2 \quad \text{for some } \mu,$$

we solve $w_{\bar z} = \mu w_z$ (Beltrami equation, w a diffeomorphism).

$$= \rho^2 \left| dz + \frac{w_{\bar z}}{w_z} d\bar{z} \right|^2$$

$$= \frac{\rho^2}{|w_z|^2} |dw|^2, \quad \text{is isothermal!}$$

So now we have



a bunch of isothermal charts.

By exercise: The transition functions are holomorphic.