From Particles to Strings

An Introduction to Quantum Field Theory

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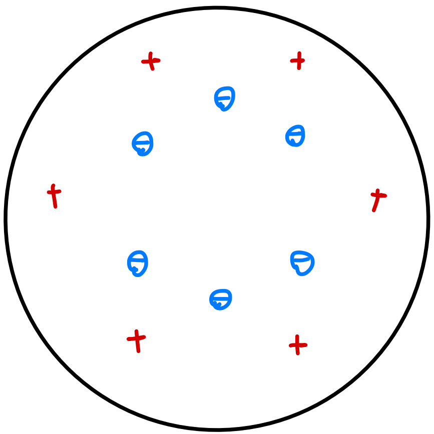
University of Denver

Geometry and Topology Seminar, University of Manitoba, 12/4/2023.

Particle Physics studies the structures of atomic and subatomic particles.

Earliest research : structure of an atom.

Thompson's Plum Pudding Model (1904).



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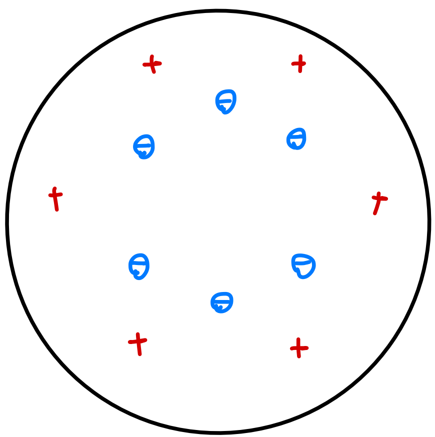
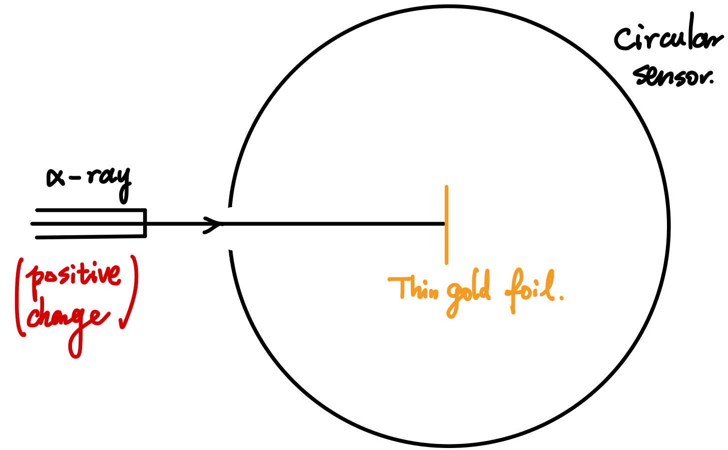


Electrons with negative charge are surrounded by a jelly-like volume of positive charge.

Rutherford's experiment (1911)



If Plum Pudding model is correct, then the high-speed α-particles should pass through w/o much deflection.

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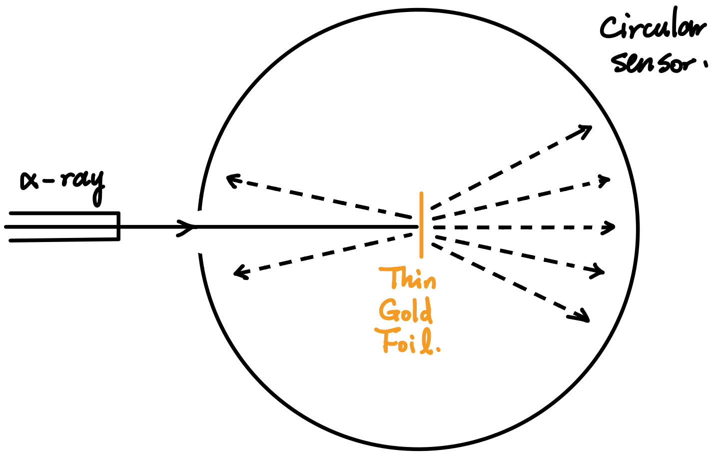

In reality, deflections occur way more than expected.

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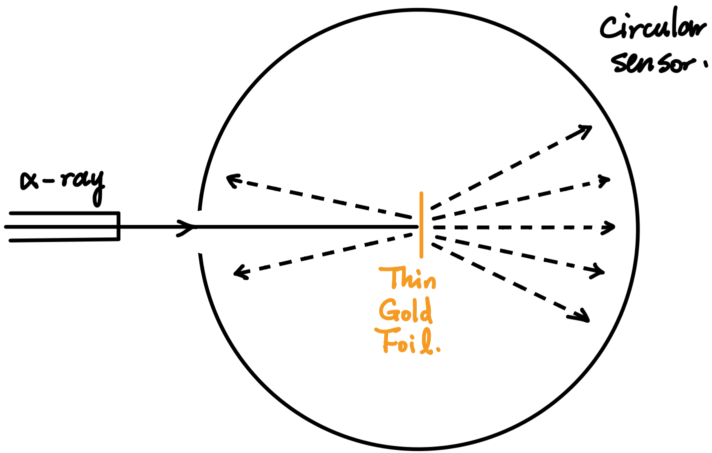
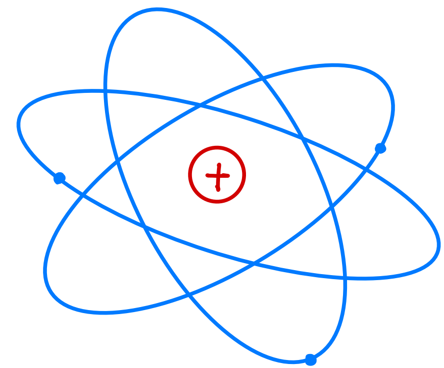
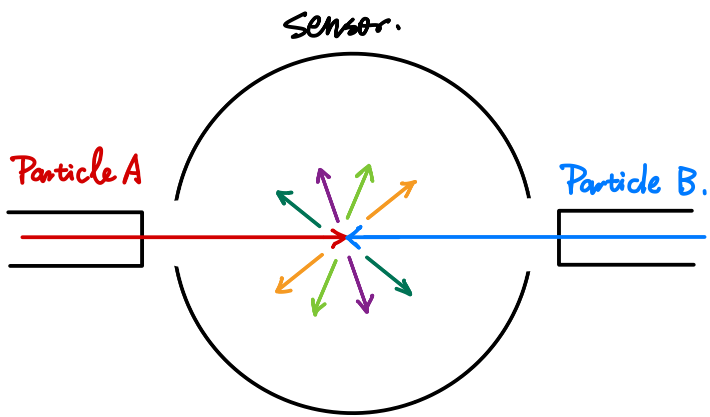




In reality, deflections occur way more than expected.

Electrons with negative charge revolve around a small nucleus with positive charge.

The method of colliding particles becomes the main experimental method.



The sensor detects the occurrence of particles generated from the process, which can be computed by scattering amplitudes of Feynman diagrams.

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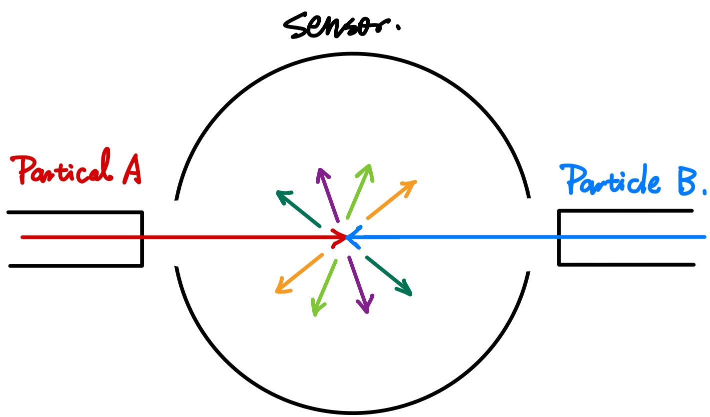

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Cross section of the Large Hadron Collider where its detectors are placed and collisions occur. Source : CERN.

Computing scattering amplitudes is the central topic in modern physics.

Phenomenological method:

Lagragian of the system

$\downarrow$

Feynmann Rules

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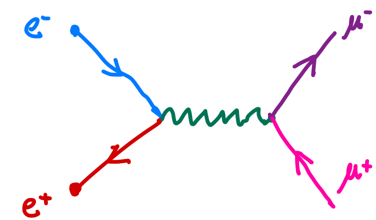
Scattering amplitudes for related Feynmann diagrams



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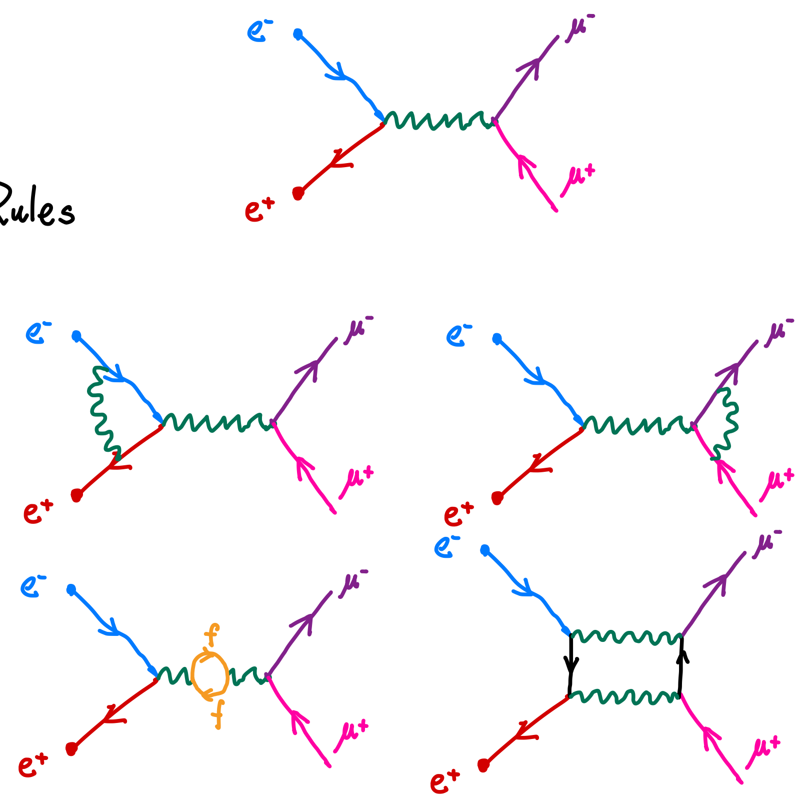
$\xRightarrow{\text{sum}}$ Occurrence of particles.

Compare with the data from collider experiments



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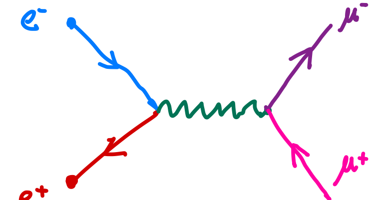
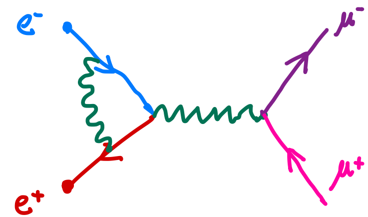
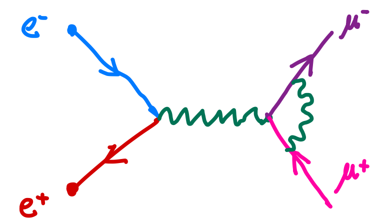
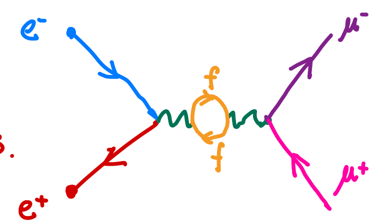
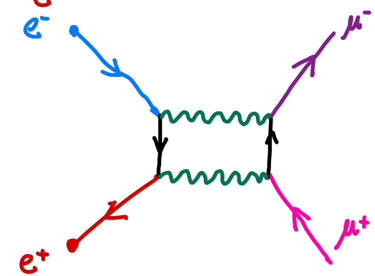










The method results in the Standard Model that unifies strong, weak & electromagnetic interactions.

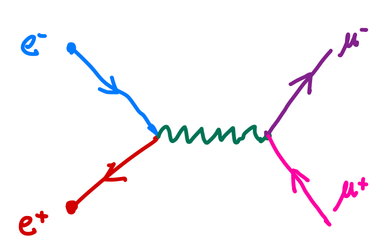
Computing scattering amplitudes is the central topic in modern physics.

Phenomenological method:

Overweight W-Bosons.

Lagragian of the system

Perturbative method
Path Integrals



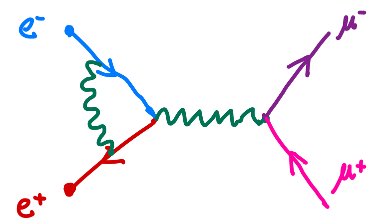
Compare with the data from collider experiments

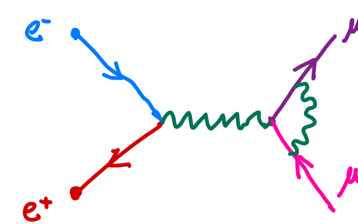
Feynmann Rules

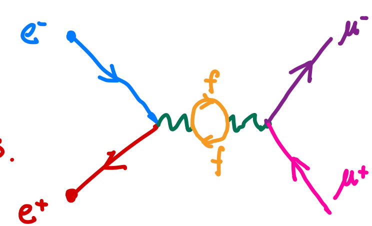
Ultraviolet Divergence.
Renormalization.

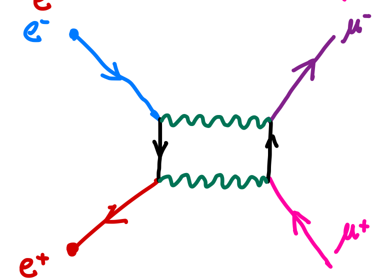
Scattering amplitudes for related Feynmann diagrams

$\overset{\text{sum}}{\Longrightarrow}$ Occurrence of particles.









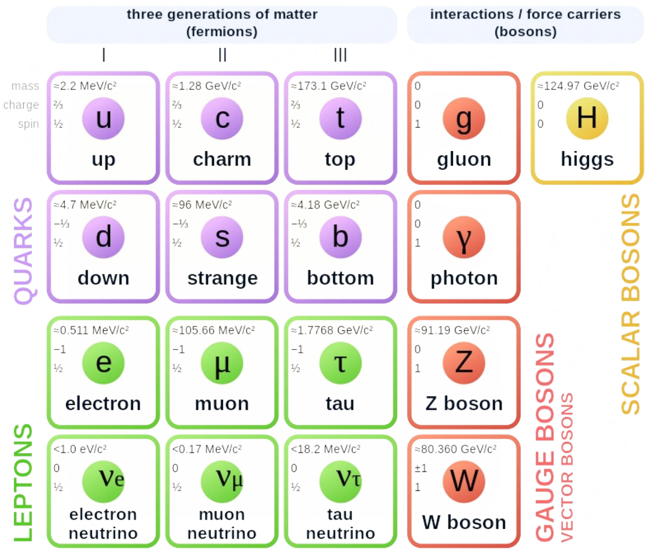
The method results in the Standard Model that unifies strong, weak & electromagnetic interactions.

But it is unsatisfactory in many aspects.

Gravity is missing

There might exist many more subatomic particles beyond Standard Model

Standard Model of Elementary Particles



The Periodic Table of the Elements

Key

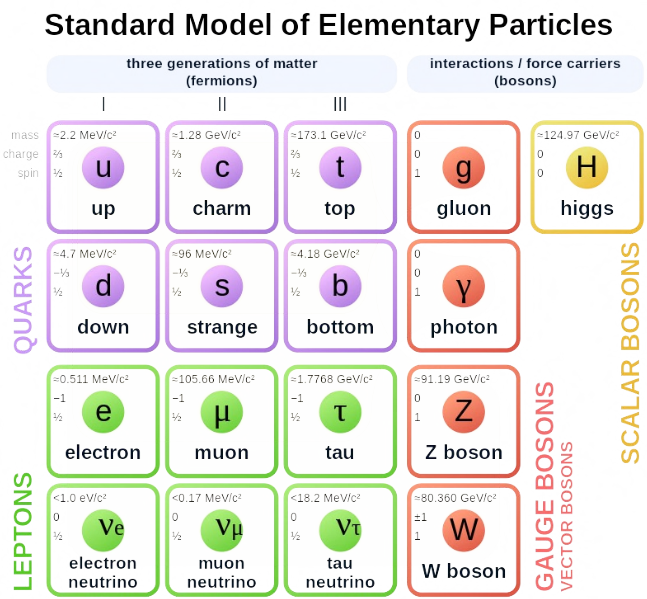
relative atomic mass
atomic symbol
name
atomic (proton) number

1	2											3	4	5	6	7	0
							1 **H** hydrogen 1										4 **He** helium 2
7 **Li** lithium 3	9 **Be** beryllium 4											11 **B** boron 5	12 **C** carbon 6	14 **N** nitrogen 7	16 **O** oxygen 8	19 **F** fluorine 9	20 **Ne** neon 10
23 **Na** sodium 11	24 **Mg** magnesium 12											27 **Al** aluminium 13	28 **Si** silicon 14	31 **P** phosphorus 15	32 **S** sulfur 16	35.5 **Cl** chlorine 17	40 **Ar** argon 18
39 **K** potassium 19	40 **Ca** calcium 20	45 **Sc** scandium 21	48 **Ti** titanium 22	51 **V** vanadium 23	52 **Cr** chromium 24	55 **Mn** manganese 25	56 **Fe** iron 26	59 **Co** cobalt 27	59 **Ni** nickel 28	63.5 **Cu** copper 29	65 **Zn** zinc 30	70 **Ga** gallium 31	73 **Ge** germanium 32	75 **As** arsenic 33	79 **Se** selenium 34	80 **Br** bromine 35	84 **Kr** krypton 36
85 **Rb** rubidium 37	88 **Sr** strontium 38	89 **Y** yttrium 39	91 **Zr** zirconium 40	93 **Nb** niobium 41	96 **Mo** molybdenum 42	[98] **Tc** technetium 43	101 **Ru** ruthenium 44	103 **Rh** rhodium 45	106 **Pd** palladium 46	108 **Ag** silver 47	112 **Cd** cadmium 48	115 **In** indium 49	119 **Sn** tin 50	122 **Sb** antimony 51	128 **Te** tellurium 52	127 **I** iodine 53	131 **Xe** xenon 54
133 **Cs** caesium 55	137 **Ba** barium 56	139 **La*** lanthanum 57	178 **Hf** hafnium 72	181 **Ta** tantalum 73	184 **W** tungsten 74	186 **Re** rhenium 75	190 **Os** osmium 76	192 **Ir** iridium 77	195 **Pt** platinum 78	197 **Au** gold 79	201 **Hg** mercury 80	204 **Tl** thallium 81	207 **Pb** lead 82	209 **Bi** bismuth 83	[209] **Po** polonium 84	[210] **At** astatine 85	[222] **Rn** radon 86
[223] **Fr** francium 87	[226] **Ra** radium 88	[227] **Ac*** actinium 89	[261] **Rf** rutherfordium 104	[262] **Db** dubnium 105	[266] **Sg** seaborgium 106	[264] **Bh** bohrium 107	[277] **Hs** hassium 108	[268] **Mt** meitnerium 109	[271] **Ds** darmstadtium 110	[272] **Rg** roentgenium 111	Elements with atomic numbers 112-116 have been reported but not fully authenticated						

****The lanthanoids (atomic numbers 58-71) and the actinoids (atomic numbers 90-103) have been omitted.**

The relative atomic masses of copper and chlorine have not been rounded to the nearest whole number.

The Lagrangian of the Standard Model is monstrous.



New approaches are necessary.

$$
\begin{gathered}
\mathbf{1}\quad -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2}ig_s^2(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
\mathbf{2}\quad M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \tfrac{1}{2}\partial_\mu H \partial_\mu H - \\
\tfrac{1}{2}m_h^2 H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \tfrac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \\
\tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - \\
W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \\
\tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + \\
g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \\
\tfrac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \\
\phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + \\
igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + \\
igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \\
\tfrac{1}{4}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) - \tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - \\
g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \\
\mathbf{3}\quad \bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \\
\tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 - \\
1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + \\
(\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 + \\
\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m_e^\lambda}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] - \\
\mathbf{4}\quad \tfrac{g}{2}\tfrac{m_e^\lambda}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + \\
m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 - \\
\gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \\
\tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j) + \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \\
\mathbf{5}\quad \tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_w W^+_\mu(\partial_\mu\bar{X}^0 X^- - \partial_\mu\bar{X}^+ X^0) + igs_w W^+_\mu(\partial_\mu\bar{Y}X^- - \\
\partial_\mu\bar{X}^+ Y) + igc_w W^-_\mu(\partial_\mu\bar{X}^- X^0 - \partial_\mu\bar{X}^0 X^+) + igs_w W^-_\mu(\partial_\mu\bar{X}^- Y - \\
\partial_\mu\bar{Y}X^+) + igc_w Z^0_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ - \\
\partial_\mu\bar{X}^- X^-) - \tfrac{1}{2}gM[\bar{X}^+ X^+ H + \bar{X}^- X^- H + \tfrac{1}{c_w^2}\bar{X}^0 X^0 H] + \\
\tfrac{1-2c_w^2}{2c_w}igM[\bar{X}^+ X^0\phi^+ - \bar{X}^- X^0\phi^-] + \tfrac{1}{2c_w}igM[\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-] + \\
igMs_w[\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-] + \tfrac{1}{2}igM[\bar{X}^+ X^+\phi^0 - \bar{X}^- X^-\phi^0]
\end{gathered}
$$

Key Ideas of String Theory:

- 0-dim particles → 1-dim strings.

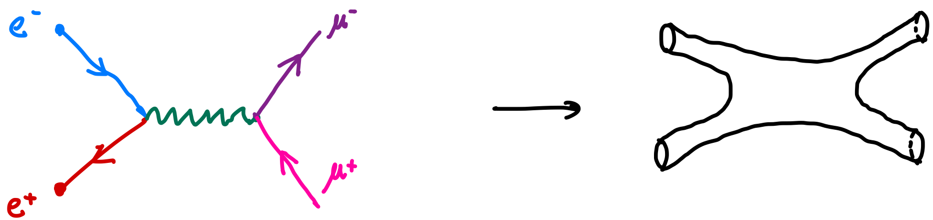
  Different particles are indeed the same string with different vibrating pattern.

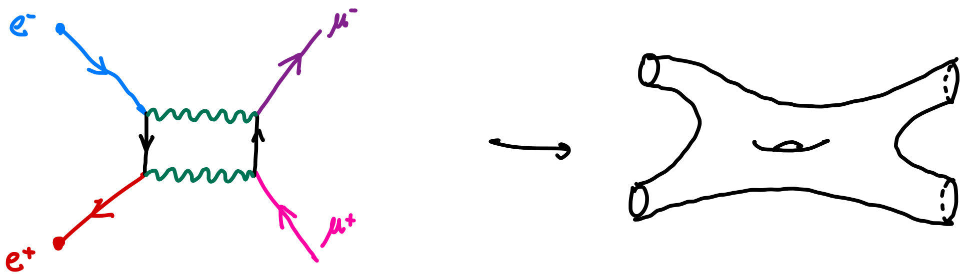
Key Ideas of String Theory :

- 0-dim particles → 1-dim strings.

  Different particles are indeed the same string with different oscillation pattern.

- Feynman diagrams → Surfaces.





While string theory is controversial in physics, it definitely leads to good mathematics.

TOPOLOGICAL QUANTUM FIELD THEORIES

by MICHAEL ATIYAH

To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten's paper [11] where he explained the geometric meaning of super-symmetry. It is well-known that the quantum Hamiltonian corresponding to a classical particle moving on a Riemannian manifold is just the Laplace-Beltrami operator. Witten pointed out that, for super-symmetric quantum mechanics, the Hamiltonian is just the Hodge-Laplacian. In this super-symmetric theory differential forms are bosons or fermions depending on the parity of their degrees. Witten went on to introduce a modified Hodge-Laplacian, depending on a real-valued function f. He was then able to derive the Morse theory (relating critical points of f to the Betti numbers of the manifold) by using the standard limiting procedures relating the quantum and classical theories.

THE DEFINITION OF CONFORMAL FIELD THEORY

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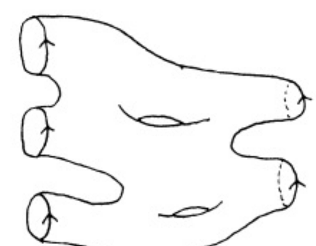
England

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by physicists.

1. THE CATEGORY $\mathcal{C}$

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n\geq 0}$, where C_n is the disjoint union of a set of n parametrized circles.

A morphism $C_n \to C_m$ is a Riemann surface X with boundary ∂X, together with an identification $i : C_m - C_n \to \partial X$. (We identify morphisms $(X,i),(X',i')$ if there is an isomorphism $f : X \to X'$ such that $f \circ i = i'$. Notice that the boundary of a Riemann surface is canonically oriented. The identifications i are supposed to be orientation-preserving, and $C_m - C_n$ means the union $C_m \amalg C_n$ with the orientation of C_n reversed.)



A morphism $C_3 \to C_2$.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

Datum: Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of $\mathbb{R}$ or S^1)

open string closed string

$\rightsquigarrow$ $Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

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$\rightsquigarrow$ $Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

__Axioms:__ (1) Z is __functorial__ w.r.t. orientation preserving diffeomorphisms of Σ and M.

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M 2d mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

<u>Axioms:</u> (1) Z is <u>functorial</u> w.r.t. orientation preserving diffeomorphisms of Σ and M.

i.e., • $f: \Sigma \to \Sigma'$, $g: \Sigma' \to \Sigma''$ orientation preserving diffeomorphisms

$\Rightarrow Z(f): Z(\Sigma) \to Z(\Sigma')$, $Z(g): Z(\Sigma') \to Z(\Sigma'')$, $Z(gf) = Z(g) Z(f)$.

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Axioms: (1) Z is functorial w.r.t. orientation preserving diffeomorphisms of Σ and M.

i.e., • $f: \Sigma \to \Sigma'$, $g: \Sigma' \to \Sigma''$ orientation preserving diffeomorphisms

$\Rightarrow$ $Z(f): Z(\Sigma) \to Z(\Sigma')$, $Z(g): Z(\Sigma') \to Z(\Sigma'')$, $Z(gf) = Z(g)Z(f)$.

• If $f: \partial M \to \partial M'$ extends to $M \to M'$,

then $Z(f): Z(\partial M) \to Z(\partial M')$ takes $Z(M)$ to $Z(M')$.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

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open string closed string

$\rightsquigarrow Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Axioms: (1) Z is functorial. (2) Z is involutory,

Atiyah's axioms for 2d TQFT over a ground ring Λ.

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open string closed string

$\rightsquigarrow Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

<u>Axioms:</u> (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>,

i.e., if Σ^* is Σ with opposite orientation,

then $Z(\Sigma^*) = Z(\Sigma)^* = \mathrm{Hom}_\Lambda(Z(\Sigma), \Lambda)$. (dual module).

Atiyah's axioms for 2d TQFT over a ground ring Λ.

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open string closed string

$\rightsquigarrow$ $Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

<u>Axioms</u>: (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>. (3) Z is <u>multiplicative</u>.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

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<u>Axioms:</u> (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>. (3) Z is <u>multiplicative</u>.

- For the disjoint union $\Sigma_1 \sqcup \Sigma_2$, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

<u>Datum:</u> Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of $\mathbb{R}$ or S^1)

open string   closed string

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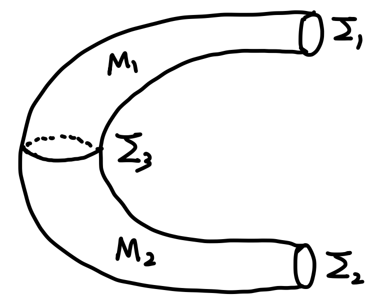
<u>Axioms:</u> (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>. (3) Z is <u>multiplicative</u>.

- For the disjoint union $\Sigma_1 \amalg \Sigma_2$, $Z(\Sigma_1 \amalg \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.
- If $\partial M_1 = \Sigma_1 \amalg \Sigma_3$, $\partial M_2 = \Sigma_2 \amalg \Sigma_3^*$, $M = M_1 \cup_{\Sigma_3} M_2$.



then $Z(M) = \langle Z(M_1), Z(M_2) \rangle$,

where $\langle\ ,\ \rangle$ is the natural pairing:

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2).$$

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M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

<u>Axioms:</u> (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>. (3) Z is <u>multiplicative</u>.

Very strong axiom.

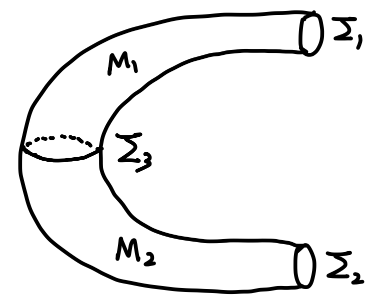
- For the disjoint union $\Sigma_1 \sqcup \Sigma_2$, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.

- If $\partial M_1 = \Sigma_1 \sqcup \Sigma_3$, $\partial M_2 = \Sigma_2 \sqcup \Sigma_3^*$, $M = M_1 \cup_{\Sigma_3} M_2$.

$Z(M)$ can be computed by "cutting M in half" along any Σ_3.

then $Z(M) = \langle Z(M_1), Z(M_2) \rangle$,

where $\langle \, , \, \rangle$ is the natural pairing:



$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2).$$

Atiyah's axioms for 2d TQFT over a ground ring Λ.

Datum: Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of $\mathbb{R}$ or S^1)

open string closed string

$\rightsquigarrow$ $Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

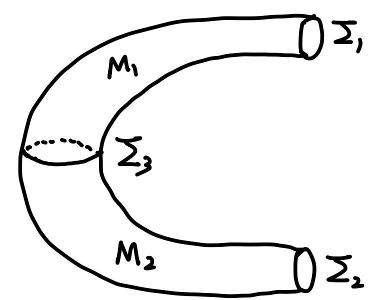
Axioms: (1) Z is functorial. (2) Z is involutory. (3) Z is multiplicative.

- For the disjoint union $\Sigma_1 \sqcup \Sigma_2$, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.
- (Equivalently) If $\partial M = \Sigma_1 \sqcup \Sigma_0^*$, then

  $Z(M) \in Z(\Sigma_0)^* \otimes Z(\Sigma_1) = \mathrm{Hom}\left(Z(\Sigma_0), Z(\Sigma_1)\right)$,

  i.e., any cobordism M between Σ_0 & Σ_1 induces

  $Z(M) : Z(\Sigma_0) \to Z(\Sigma_1)$.



We require that this is transitive when we compose cobordisms.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

Datum: Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of $\mathbb{R}$ or S^1)

open string closed string

$\rightsquigarrow Z(\Sigma)$ fin. gen. Λ-module.

M 2d mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Axioms: (1) Z is functorial. (2) Z is involutory. (3) Z is multiplicative.

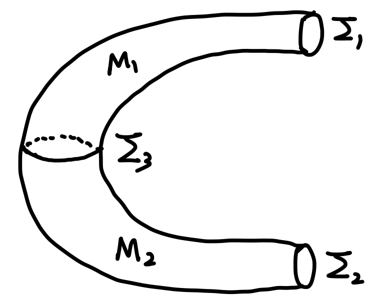
- For the disjoint union $\Sigma_1 \sqcup \Sigma_2$, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.
- (Equivalently) If $\partial M = \Sigma_1 \sqcup \Sigma_0^*$, then

Z forms a functor from the cobordism category to Λ-mod category.

$$Z(M) \in Z(\Sigma_0)^* \otimes Z(\Sigma_1) = \mathrm{Hom}\left(Z(\Sigma_0), Z(\Sigma_1)\right),$$

i.e., any cobordism M between Σ_0 & Σ_1 induces

$$Z(M) : Z(\Sigma_0) \to Z(\Sigma_1).$$



We require that this is transitive when we compose cobordisms.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

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M 2d mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Axioms: (1) Z is functorial. (2) Z is involutory. (3) Z is multiplicative.

(4) Non-triviality: $\cdot\ \Sigma = \phi,\ Z(\Sigma) = \Lambda$ $\quad\cdot\ M = \phi,\ Z(M) = 1.$

$\cdot\ Z(\Sigma \times I) : Z(\Sigma) \to Z(\Sigma)$ is the identity

Atiyah's axioms for 2d TQFT over a ground ring Λ.

<u>Datum:</u> Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of $\mathbb{R}$ or S^1)

$\rightsquigarrow$ $Z(\Sigma)$ fin. gen. Λ-module, with a nondeg. Hermitian structure. So that $Z(\Sigma^*) = \overline{Z(\Sigma)}$.

M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

<u>Axioms:</u> (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>. (3) Z is <u>multiplicative</u>.

(4) <u>Non-triviality</u>. (5) <u>Hermitian</u>:

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M 2d mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

__Axioms:__ (1) Z is __functorial__. (2) Z is __involutory__. (3) Z is __multiplicative__.

(4) __Non-triviality__. (5) __Hermitian__: Let M^* be M with reverse orientation.

Then $Z(M^*) = \overline{Z(M)}$.

Atiyah's axioms for 2d TQFT over a ground ring Λ.

<u>Datum:</u> Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of $\mathbb{R}$ or S^1)

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M 2d mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

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(4) <u>Non-triviality</u>. (5) <u>Hermitian</u>: Let M^* be M with reverse orientation.

Then $Z(M^*) = \overline{Z(M)}$.

(Equivalently) If $\partial M = \Sigma_0^* \sqcup \Sigma_1$, $Z(M) : Z(\Sigma_0) \to Z(\Sigma_1)$,

then $Z(M^*)$ is the adjoint of $Z(M)$.

Atiyah's axioms for ~~2d~~ TQFT over a ground ring Λ.

Datum: Σ $\cancel{1d}$ d-dim oriented closed smooth mnfd ~~(homeo. to disjoint unions of $\mathbb{R}$ or S^1)~~

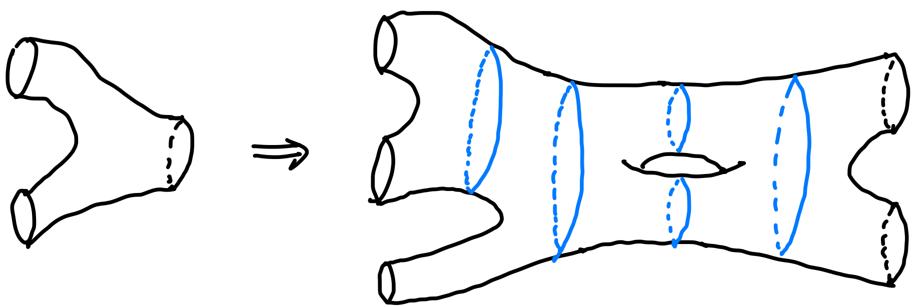
$\rightsquigarrow$ $Z(\Sigma)$ fin. gen. Λ-module, with a nondeg. Hermitian structure. So that $Z(\Sigma^*) = \overline{Z(\Sigma)}$.

M $\cancel{2d}$ (d+1)-dim mnfd with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Axioms: (1) Z is functorial. (2) Z is involutory. (3) Z is multiplicative.

(4) Non-triviality. (5) Hermitian.

Rmk: (3) and (5) $\Rightarrow$ a 2d TQFT is determined by the Z-image of pants.



Analogous to Feynman Rules at the vertices.

Atiyah's axioms for ~~2d~~ TQFT over a ground ring Λ.

Datum: Σ d-dim ~~1d~~ oriented closed smooth mnfd ~~(homeo. to disjoint unions of $\mathbb{R}$ or S^1)~~

$\rightsquigarrow Z(\Sigma)$ fin. gen. Λ-module, with a nondeg. Hermitian structure. So that $Z(\Sigma^*) = \overline{Z(\Sigma)}$.

M (d+1)-dim. ~~2d~~ mnfd with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Axioms: (1) Z is functorial. (2) Z is involutory. (3) Z is multiplicative.

(4) Non-triviality. (5) Hermitian.

Physically: Σ indicates the physical space.

$Z(\Sigma)$ Hilbert space of quantum theory

If $\partial M = \Sigma$, then $Z(M) \in Z(\Sigma)$ is the vacuum state defined by M.

If M is closed, then $Z(M)$ is the vacuum-vacuum expectation value,

aka, partition function in Statistical physics (can be checked by experiments).

Segal's axioms for 2d closed string CFT

Datum: Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of ~~$\mathbb{R}$~~ or S^1)

$\rightsquigarrow Z(\Sigma) = H^{\otimes m}$ $m = \#$copies of S^1, H complex Hilbert space.

M Riemann surface with bdry $\partial M \rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

Axioms: (1) Z is functorial. (2) Z is involutory. (3) Z is multiplicative.

(4) Non-triviality. (5) Hermitian. (With analytic modifications)

(6) Collapsing Property (Convergence of traces of $Z(M)$).

Segal's axioms for 2d closed ring CFT

<u>Datum:</u> Σ 1d oriented closed smooth mnfd (homeo. to disjoint unions of ~~$\mathbb{R}$~~ or S^1)

$\rightsquigarrow$ $Z(\Sigma) = H^{\otimes m}$ $m = \#$copies of S^1, H complex Hilbert space.

M Riemann surface with bdry ∂M $\rightsquigarrow$ Element $Z(M) \in Z(\partial M)$.

<u>Axioms:</u> (1) Z is <u>functorial</u>. (2) Z is <u>involutory</u>. (3) Z is <u>multiplicative</u>.

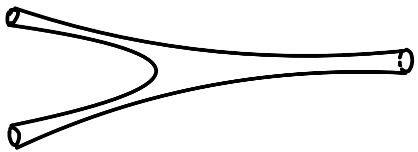
(4) <u>Non-triviality</u>. (5) <u>Hermitian</u>. (With analytic modifications)

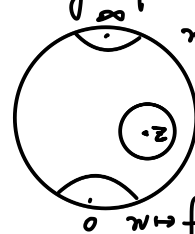
(6) <u>Collapsing Property</u> (Convergence of traces of $Z(M)$).

<u>Rmks:</u> There exists many examples of 2d TQFT (finite-dimensional Frobenius algebra) and "weak" 2d CFT (vertex algebras $\rightsquigarrow$ Intw. op. alg. $\rightsquigarrow$ Full field algebra).

The full construction of 2d CFT is both extremely important & extremely difficult

Under certain assumptions, a pant is conformally equivalent to a Riemann sphere with 3 punctures





$w \mapsto f_\infty(\frac{1}{w})$

$w \mapsto f_z(w)$

$w \mapsto f_0(w)$

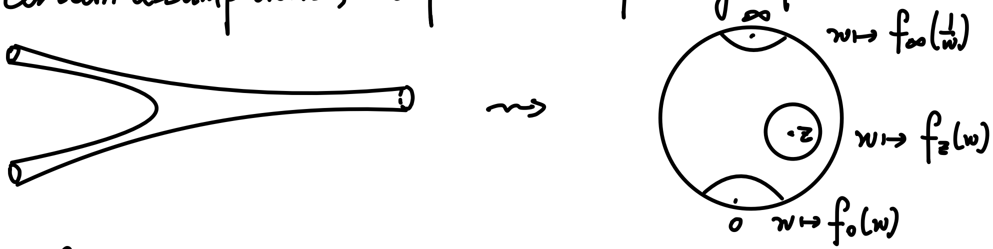
Simplest Case:

$f_\infty(\frac{1}{w}) = \frac{1}{w}$.

$f_z(w) = w - z$

$f_0(w) = w$.

Under certain assumptions, a pant is conformally equivalent to a Riemann sphere with 3 punctures



Simplest Case:

$f_\infty\left(\frac{1}{w}\right) = \frac{1}{w}$.

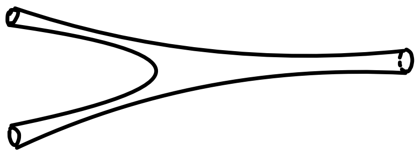
$f_z(w) = w - z$

$f_0(w) = w$.

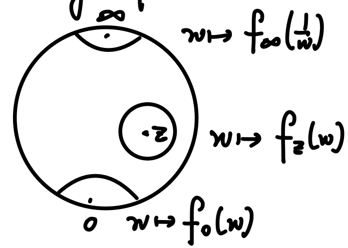
The Z-functor in a 2d CFT should define a linear map $Y(\cdot, z) : H \otimes H \to H$ that is

- Dependent on z and the local coordinates f_0, f_z, f_∞. (necessarily, $\dim H = \infty$).

Under certain assumptions, a pant is conformally equivalent to a Riemann sphere with 3 punctures



$\rightsquigarrow$



Simplest Case:

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The Z-functor in a 2d CFT should define a linear map $Y(\cdot, z) : H \otimes H \to H$ that is

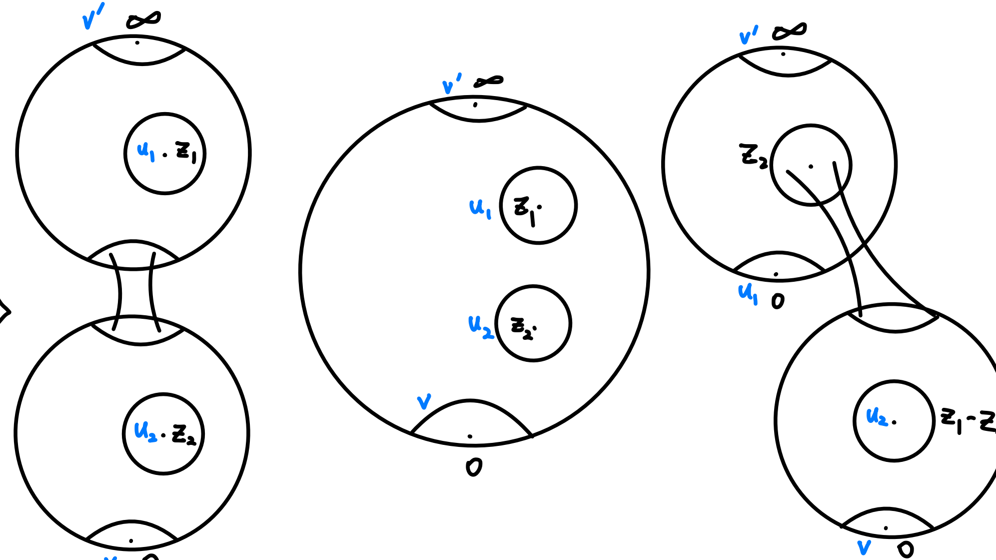
- Dependent on z and the local coordinates f_0, f_z, f_∞. (necessarily, $\dim H = \infty$).
- Satisfy associativity property:

$$\langle v', Y(u_1, z_1) Y(u_2, z_2) v \rangle$$

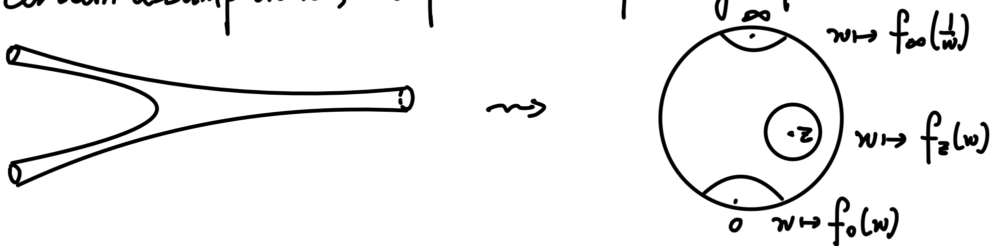
$$= \langle v', Y(Y(u_1, z_1 - z_2) u_2, z_2) v \rangle$$

$$|z_1| > |z_2| > |z_1 - z_2| > 0$$

Analogous to $a(bc) = (ab)c$



Under certain assumptions, a pant is conformally equivalent to a Riemann sphere with 3 punctures



Simplest Case:

$f_\infty(\frac{1}{w}) = \frac{1}{w}$.

$f_z(w) = w - z$

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The Z-functor in a 2d CFT should define a linear map $Y(\cdot, z) : H \otimes H \to H$ that is

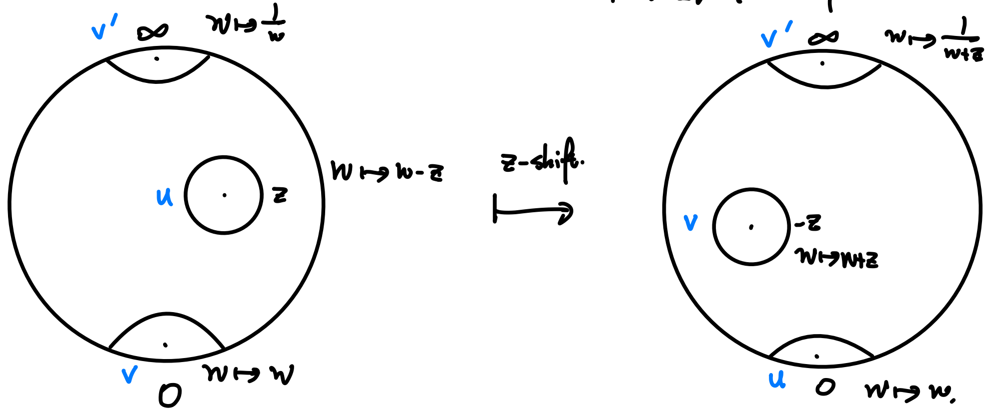
- Dependent on z and the local coordinates f_0, f_z, f_∞. (necessarily, $\dim H = \infty$).
- Satisfy associativity property: $\langle v', Y(u_1, z_1) Y(u_2, z_2) v \rangle = \langle v', Y(Y(u_1, z_1 - z_2) u_2, z_2) v \rangle$

  $|z_1| > |z_2| > |z_1 - z_2| > 0$.
- Conformally equivalent.

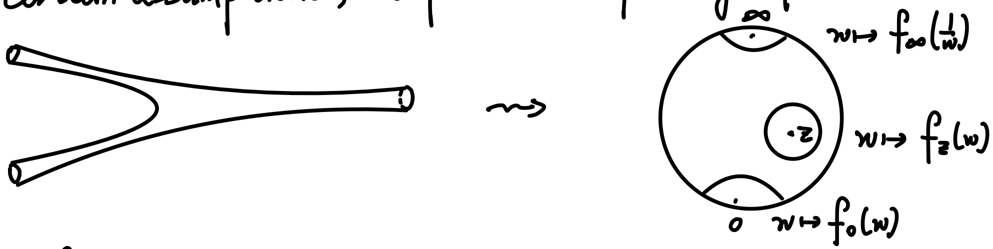
$\langle v', Y(u, z) v \rangle$

$= \langle v', e^{zL(-1)} Y(v, -z) u \rangle$, $|z| > 0$.

Analogous to $ab = ba$.



Under certain assumptions, a pant is conformally equivalent to a Riemann sphere with 3 punctures



Simplest Case:

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$f_0(w) = w$.

The Z-functor in a 2d CFT should define a linear map $Y(\cdot, z) : H \otimes H \to H$ that is

- Dependent on z and the local coordinates f_0, f_z, f_∞. (necessarily, $\dim H = \infty$).
- Satisfy multiplicative property: $\langle v', Y(u_1, z_1) Y(u_2, z_2) v \rangle = \langle v', Y(Y(u_1, z_1 - z_2) u_2, z_2) v \rangle$
  $|z_1| > |z_2| > |z_1 - z_2| > 0$.
- Conformally equivalent. $\langle v', Y(u, z) v \rangle = \langle v', e^{zL(-1)} Y(v, -z) u \rangle$, $|z| > 0$.

There are many examples of such Y and H (with $\dim H$ countably infinite)

- If $\langle v', Y(u, z_1) Y(v, z_2) v \rangle$ is a rational function $\rightsquigarrow$ Vertex algebras.
- If $\langle v', Y(u, z_1) Y(v, z_2) v \rangle$ is a (multivalued) holomorphic function $\rightsquigarrow$ Intw. Op. Algebra
- Full field algebra = Intw Op. algebra + Its antiholomorphic part.

Brief history of vertex algebras, intertwining operator algebras & full field algebras.

- Lepowsky-Wilson (1979). Free field realization of affine Lie algebras.
- Borcherds (1986). Frenkel-Lepowsky-Meurman (1989).

  Moonshine module for the Monster Group.
- Frenkel-Huang-Lepowsky (1993). Axiomatic Approach to Vertex Algebras.
- Huang-Lepowsky (1994). (1999) Huang (2007)

  Vertex tensor category structure, Intertwining Operator Algebra,

  Convergence of traces (genus-1 rational C_2-cofinite CFT)
- Gui (2020) Convergence of multitraces (higher-genus rational C_2-cofinite CFT).
- Huang-Kong (2007). Open-string Vertex Algebras. Full Field Algebras.