ORDERABLE GROUPS MINICOURSE NOTES

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ABSTRACT. These minicourse notes serve as an introduction to the various kinds of total orders on groups, and their behaviour with respect to other standard algebraic structures and constructions, such as quotients, free products, and amalgams. They are parepared presuming no prior knowledge of group orderings and develop many of the essential tools and structures from scratch, such as the dynamical realisation of an ordering, the Burns-Hale theorem, Hölder's theorem and its relationship with local indicability, and so forth. Many of the topics, even at a basic level, lead naturally to open problems in the field, which are mentioned throughout the text either in the form of a question or a conjecture.

1. LECTURE 1: LEFT-ORDERINGS AND BI-ORDERINGS

Definition 1.1. *A* left-ordering *of a group G is a strict total ordering $<$ of the elements of G such that*

$$g < h \Rightarrow fg < fh$$

for all $f, g, h \in G$. A bi-ordering *of G is a left-ordering that also satisfies*

$$g < h \Rightarrow gf < hf$$

for all $f, g, h \in G$.

A group equipped with a specified left order bi-ordering will be called an *ordered group* and written as a pair $(G, <)$. A group which admits a left-ordering (resp. bi-ordering) will be called a left-orderable group (resp. bi-orderable group). We'll write LO group and BO group for short.

There is an alternative characterisation.

Definition 1.2. *A group G is LO if there exists a subset $P \subset G$ satisfying*

(1) $P \cdot P \subset P$,
(2) $G \setminus \{id\} = P \sqcup P^{-1}$.

A subset P satisfying these two properties is called a positive cone.

There's a correspondence between positive cones and orderings on G via

$$< \quad \mapsto \quad \{g \in G \mid g > id\}$$

and

$$P \quad \mapsto \quad g < h \text{ if and only if } g^{-1}h \in P.$$

One can check that this defines a bijection. A group is BO if it admits a $P \subset G$ satisfying (1) and (2) above, and also (3) $gPg^{-1} \subset P$ for all $g \in G$.

Example 1.3. With only the definition in hand, examples are tricky to come by. Obviously $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all BO groups with equipped with addition. $\square$

Example 1.4. We can order $\mathbb{Z}^2$ by choosing $(x_1, x_2) \in \mathbb{R}^2$ such that x_1/x_2 is irrational, and declaring $(m, n) > (0, 0)$ if and only if $mx_1 + nx_2 > 0$ for all $(m, n) \in \mathbb{Z}^2$. This corresponds to taking a line of irrational slope passing through the origin, and declaring all elements on one side of the line to be the positive cone, see Figure 1. This obviously generalizes to $\mathbb{Z}^n$, though you you have to take care to choose a hyperplane that avoids all the integer lattice points in $\mathbb{R}^n$. $\square$

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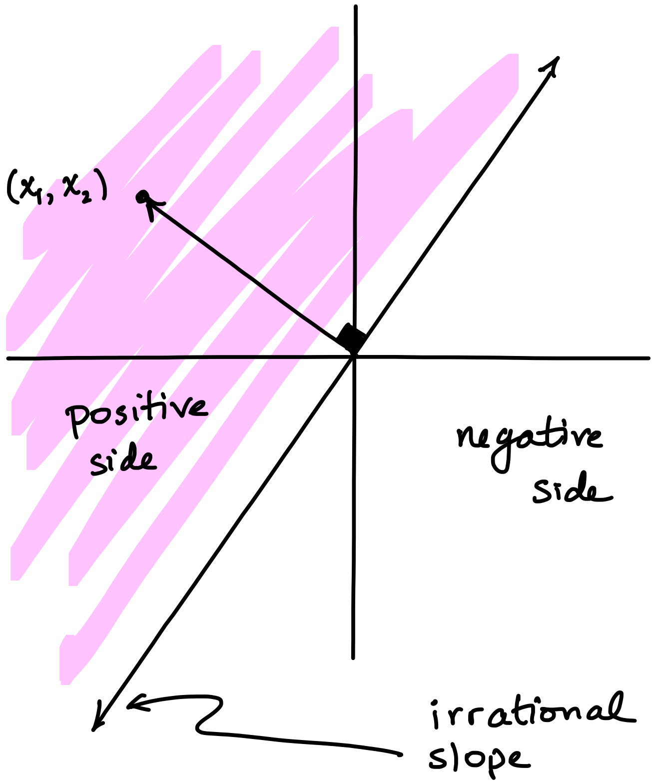

FIGURE 1. Ordering $\mathbb{Z}^2$ using a vector of irrational slope.

Getting fancier examples than this without having any sophisticated tools in hand requires a bit of cleverness, so let's see one such example.

Example 1.5. (Due to Magnus, following [9, Chapter 3]) We show in this example that the free group F on countably many generators $\{x_1, x_2, \dots\}$ is bi-orderable, so all of the finitely generated free groups are, too. Set

$$\Lambda = \mathbb{Z}[[X_1, X_2, \dots]],$$

the ring of formal power series in non-commuting variables. Define $\mu : F \to \Lambda$ by

$$\mu(x_i) = 1 + X_i, \text{ and } \mu(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots.$$

So, for example

$$\mu(x_1x_2) = (1 + X_1)(1 + X_2) = 1 + X_1 + X_2 + X_1X_2,$$

or one can check also that, no matter if $p > 0$ or $p < 0$, one always has

$$\mu(x_i^p) = 1 + pX_i + O(2)$$

where $O(2)$ is terms of degree two and higher. Then we observe two lemmas that together complete the proof:

Lemma 1.6. *Let G denote the subgroup of Λ consisting of elements of the form $1 + O(1)$. Then G is bi-orderable.*

Proof. Write the elements of G with lowest degree terms first, and in each degree, order the terms lexicographically (in fact, any fixed ordering in each degree will do). Then, if $U, V \in \Lambda$, declare $U < V$ if the first coefficients where U, V differ satisfy this same inequality. E.g. if

$$U = 1 + X_1 + X_2 + 3X_1^2 + \dots \text{ and } V = 1 + X_1 + X_2 + 5X_1^2 + \dots$$

then $U < V$ because $3 < 5$. From here it is a straightforward check to verify that this works. $\square$

Lemma 1.7. *The homomorphism $\mu : F \to \Lambda$ is injective.*

Proof. One checks that this is true by showing that if $w = x_{i_1}^{n_1} \cdots x_{i_k}^{n_k}$ then the coefficient of the term $X_{i_1}^{n_1} \cdots X_{i_k}^{n_k}$ in the expression for $\mu(w)$ is p, in particular, $\mu(w) \neq 1$. $\square$

We can also create plenty of left-orderable groups using extensions, as this requires little more than the definition.

Proposition 1.8. *Suppose that $P_K \subset K$ and $P_H \subset H$ are positive cones (so that K and H are LO), and that*

$$\{id\} \to K \xrightarrow{i} G \xrightarrow{q} H \to \{id\}$$

is a short exact sequence. Then $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone, in particular, G is LO.

Proof. Check the definition. $\square$

Example 1.9. Torsion-free metabelian groups with torsion-free abelianization are left-orderable. For example, the Heisenberg group over $\mathbb{R}$ is the group of matrices

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \ : \ a, b, c \in \mathbb{R} \right\}$$

is left-orderable for this reason. The group

$$K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$$

is also left-orderable by the same argument, since there is a short exact sequence

$$\{id\} \to \mathbb{Z} \xrightarrow{i} K \xrightarrow{q} \mathbb{Z} \to \{id\}.$$

Note, however, that the two ends of this short exact sequence are BO groups, while the centre is clearly not BO. $\square$

Proposition 1.10. *If $P_K \subset K$ and $P_H \subset H$ are positive cones of bi-orderings, and*

$$\{id\} \to K \xrightarrow{i} G \xrightarrow{q} H \to \{id\}$$

is a short exact sequence, show that $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone if and only if $gi(P_K)g^{-1} \subset i(P_K)$ for all $g \in G$.

Proof. Check the definition. $\square$

We need additional tools to produce more examples of LO and BO groups, aside from these few. The next theorem characterizes such groups completely. Recall first that a G-action on a set X is *effective* if $g \cdot x = x$ for all $x \in X$ implies $g = id$.

Theorem 1.11. *A group is left-orderable if and only if it admits an effective action by order-preserving bijections on a totally ordered set.*

Proof. If G is LO, first fix a left-ordering $<$ of G and then set $(X, <) = (G, <)$. Then G acts on X by left-multiplication, which is clearly and order-preserving effective action by bijections.

On the other hand, suppose that $(X, <)$ has an effective, order-preserving G-action. Choose a well-order $\prec$ of X (completely unrelated to the ordering $<$ of X!) and for every $g \in G \setminus \{id\}$, set

$$x_g = \min_{\prec} \{x \in X \mid g \cdot x \neq x\}.$$

Note x_g exists because the action is effective. Now, define a positive cone $P \subset G$ by $g \in P$ if and only if $g \cdot x_g > x_g$.

To check this works, it is straightforward to see that $P \sqcup P^{-1} = G \setminus \{id\}$. Then, given $g, h \in P$ suppose that $x_g \prec x_h$, the case of $x_h \prec x_g$ being similar. Observe that $x_{gh} = x_g$, because $h \cdot x_g = x_g$ and so $gh \cdot x_g = g \cdot x_g \neq x_g$; while $g \cdot x = x$ and $h \cdot x = x$ for all $x \prec x_g$. So we compute that

$$gh \cdot x_{gh} = gh \cdot x_g = g \cdot x_g > x_g = x_{gh},$$

so that $gh \in P$. □

Proposition 1.12. *A group G is bi-orderable if and only if it acts effectively by order-preserving bijections on a totally ordered set $(X, <)$, and further*

$$\forall g \in G[(\exists x \text{ s.t. } g \cdot x > x) \Rightarrow (g \cdot x \geq x \text{ for all } x \in X)].$$

When G is a countable group, these results can be greatly improved in a way that connections LO and BO groups to dynamics.

Theorem 1.13. *Suppose that G is countable. Then G is LO if and only if there exists an embedding $G \to \mathrm{Homeo}_+(\mathbb{R})$.*

Proof. The "$\Leftarrow$" direction is already clear from the previous theorem, but "$\Rightarrow$" requires a construction, see e.g. [22].

First, define a *gap* in $(G, <)$ to be a pair of elements (g, h) with $g < h$ such that there is no $f \in G$ with $g < f < h$. Then call an order-preserving embedding $t : (G, <) \to (\mathbb{R}, <)$ *tight* if $(a, b) \subset \mathbb{R} \setminus t(G)$ implies that $(a, b) \subset (t(g), t(h))$ for some gap (g, h) in $(G, <)$–i.e., the only gaps in the image of t come from gaps in G.

Tight embeddings exist whenever G is countable, for any ordering $<$ of G. To see this, we enumerate $G = \{g_0 = id, g_1, g_1, \dots\}$ and set $t(id) = 0$. Then if $t(id), t(g_1), \dots, t(g_k)$ are already defined, we set:

$$t(g_{k+1}) = \begin{cases} \max\{t(g_0), \dots, t(g_k)\} + 1 & \text{if } g_{i+1} > \max\{g_0, \dots, g_k\} \\ \min\{t(g_0), \dots, t(g_k)\} - 1 & \text{if } g_{k+1} < \min\{g_0, \dots, g_k\} \\ \dfrac{t(g_j) + t(g_i)}{2} & \text{if } g_j < g_{k+1} < g_i \text{ and} \\ & \nexists \ell \in \{0, \dots, k\} \text{ s.t. } g_j < g_\ell < g_i. \end{cases}$$

One can verify that this is tight. Then given a tight $t : (G, <) \to (\mathbb{R}, <)$ we can build $\rho : G \to \mathrm{Homeo}_+(\mathbb{R})$ via:

(1) If $x \in t(G)$ then $x = t(h)$ for some $h \in G$ and set $\rho(g)(t(h)) = t(gh)$,
(2) if $x \in \overline{t(G)}$, then define $\rho(g)(x)$ so that $\rho(g)$ is continuous on $\overline{t(G)}$, e.g. using sequences,
(3) if $x \in \mathbb{R} \setminus \overline{t(G)}$, then there exists a gap h, k such that $x \in (t(h), t(k))$. Write

$$x = (1 - s)t(h) + st(k)$$

for some $s \in (0, 1)$ and define

$$\rho(g)(x) = (1 - s)t(gh) + st(gk).$$

It is a long check, but this works, and defines a *dynamic realisation* of $(G, <)$. □

One can check that dynamic realisations, as defined in the previous proof, are unique up to conjugation.

Proposition 1.14. *Any two dynamic realisations of $(G, <)$ are conjugate. In other words, given tight embeddings $t, t' : (G, <) \to (\mathbb{R}, <)$ and corresponding dynamic realisations $\rho, \rho' : G \to \mathrm{Homeo}_+(\mathbb{R})$, there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that*

$$\rho(g)(x) = h \circ \rho(g) \circ h^{-1}(x)$$

for all $g \in G$ and $x \in \mathbb{R}$.

Remark 1.15. This means that the collection of countable, LO groups is precisely the collection of groups that embed into $\mathrm{Homeo}_+(\mathbb{R})$. Moreover, the study of orderings of such G is equivalent to the study of certain embeddings (up to conjugation) of G into $\mathrm{Homeo}_+(\mathbb{R})$.

Example 1.16. The Baumslag-Solitar groups $B(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$ are all left-orderable, and we can exhibit an affine action that demonstrates why in the case where $m = 1$. Define

$$\phi : BS(1,n) \to \mathrm{Homeo}_+(\mathbb{R})$$

by $\phi(a)(x) = nx$ and $\phi(b)(x) = x + 1$ for all $x \in \mathbb{R}$. One can check that this is an embedding, so that $BS(1,n)$ is LO.

In fact, if $n > 0$ then $BS(1,n)$ is BO, but this action does *not* satisfy $\phi(g)(x) > x$ for some $x \in \mathbb{R}$ implies $\phi(g)(x) > x$ for all $x \in \mathbb{R}$, so it's not a dynamic realisation of any bi-ordering. □

So far we have given several conditions that a group may satisfy in order to be LO. Next we focus on conditions satisfied by finite subsets of a given group G that can be used to show G is LO.

First, we introduce a property that a semigroup $P \subset G$ can have, which we will call property $(*)$:

For every finite set $\{g_1, \dots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin sg(P \setminus \{id\}, g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n})$.

Here, we use $sg(S)$ to denote the subsemigroup of G generated by $S \subset G$.

Theorem 1.17. *([18, Lemma 3.1.1]) Given a semigroup $Q \subset G$, there exists a positive cone $P \subset G$ with $Q \setminus \{id\} \subset P$ if and only if Q satisfies $(*)$.*

Proof. Given $Q \subset G$ with $Q \setminus \{id\}$ contained in some positive cone, it's clear that $(*)$ holds, by choosing ϵ_i so that $g_i^{\epsilon_i} \in P$.

On the other hand, suppose that Q satisfies $(*)$. First observe that if $g \in G \setminus \{id\}$, then one of $sg(Q \setminus \{id\}, g)$ or $sg(Q \setminus \{id\}, g^{-1})$ must satisfy $(*)$. For if not, then there exist $h_1, \dots, h_n$ and $f_1, \dots, f_m$ such that

$$id \in sg(Q \setminus \{id\}, g, h_1^{\epsilon_1}, \dots, h_n^{\epsilon_n})$$

no matter the choice of ϵ_i's, and

$$id \in sg(Q \setminus \{id\}, g^{-1}, f_1^{\nu_1}, \dots, f_m^{\nu_m})$$

no matter the choice of ν_i's. But then

$$id \in sg(Q \setminus \{id\}, g^{\epsilon}, h_1^{\epsilon_1}, \dots, h_n^{\epsilon_n}, f_1^{\nu_1}, \dots, f_m^{\nu_m})$$

no matter the choice of ϵ, ϵ_i's, and ν_i's, contradicting that Q satisfies $(*)$.

So now we set

$$M = \{\text{semigroups } P \subset G \text{ with } Q \subset P \text{ that satisfy } (*)\}.$$

The set M is nonempty, since it contains Q, it is partially ordered by inclusion and one can check that every chain has an upper bound simply by taking unions. So, we can choose $P \in M$ maximal.

Now $(*)$ forces $P \cap P^{-1} \subset \{id\}$, and maximality forces $G \setminus \{id\} \subset P \cup P^{-1}$. So $P \setminus \{id\}$ is the positive cone of a left-ordering of G. □

Corollary 1.18. *A group G is LO if and only if for all $\{g_1, \dots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin sg(g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n})$.*

Proof. Take $Q = \{id\}$ in the previous theorem. □

Corollary 1.19. *A group G is LO if and only if all of its finitely generated subgroups are LO.*

Corollary 1.20. *All torsion-free abelian groups are BO.*

In fact, it is good enough to consider quotients of all finitely generated subgroups in order to determine whether or not G is LO.

Theorem 1.21 (Burns-Hale, [6])**.** *A group G is LO if and only if for every finitely generated $H \leq G$, there exists a surjection $H \to L$ where L is a nontrivial LO group.*

Proof. We apply Corollary 1.18, showing by induction that we can always find the necessary ϵ_i's.

First note that for all $g \in G \setminus \{id\}$, $id \notin sg(g)$ since there exists a surjection $\langle g \rangle \to \mathbb{Z}$.

Now suppose that for all $\{g_1, \ldots, g_n\} \subset G \setminus \{id\}$ with $n \leq k$, there exists $\epsilon_i = \pm 1$ such that $id \notin sg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$. Consider a collection of elements $\{h_1, \ldots, h_{k+1}\} \subset G \setminus \{id\}$.

Set $H = \langle h_1, \ldots, h_{k+1} \rangle$ and choose $\phi : H \to L$ where L is a nontrivial LO group with positive cone $P_L \subset L$. Assume the h_i's are indexed so that $\phi(h_i) = id$ for $i = 1, \ldots, r$ and $\phi(h_i) \neq id$ for $i = r+1, \ldots, k+1$. Note there is at least one h_i such that $\phi(h_i) \neq id$, since ϕ is a surjection.

Now choose exponents $\epsilon_{r+1}, \ldots, \epsilon_{k+1}$ so that $\phi(h_i) \in P_L$ for $i = r+1, \ldots, k+1$, and by induction, choose $\epsilon_1, \ldots, \epsilon_r$ such that $id \notin sg(h_1^{\epsilon_1}, \ldots, h_r^{\epsilon_r})$.

Given $w \in sg(h_1^{\epsilon_1}, \ldots, h_{k+1}^{\epsilon_{k+1}})$, if w contains any occurences of $h_{r+1}, \ldots, h_{k+1}$ then $\phi(w) \in P_L$ and so $w \neq id$. If w contains no such occurences then $w \in sg(h_1^{\epsilon_1}, \ldots, h_r^{\epsilon_r})$ and $w \neq id$ by induction. Therefore we have found the exponents needed to apply Corollary 1.18. $\square$

We've just spent some time focused solely on LO groups, and it is fair to ask at this point if there are bi-orderability analogs of the ideas above that characterize left-orderability in terms of finite subsets.

Theorem 1.22 (Fuchs [14])**.** *A group G is BO if and only if for all $\{g_1, \ldots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin nsg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$, where $nsg(S)$ is the normal subsemigroup of G generated by S.*

It's not as straightforward as in the previous case, but this also leads to:

Theorem 1.23. *A group G is BO if and only if every finitely generated subgroup of G is BO.*

This allows us to tidy up a few earlier arguments that depended on the cardinality of certain generating sets:

Corollary 1.24. *All free groups are BO.*

Despite this, there is no BO version of the Burns-Hale theorem.

Example 1.25. The group $K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ satisfies: For all finitely generated $H \leq K$, there exists a surjection $H \to \mathbb{Z}$. Despite these maps onto BO quotients, K is clearly not BO. $\square$

With these many techniques, it is perhaps not surprising that LO groups often admit many possible left-orderings. In fact, G only admits finitely many left-orderings when it is a *Tararin group*, meaning G admits a rational series

$$T_0 = \{id\} \triangleleft T_1 \triangleleft \cdots \triangleleft T_{k-1} \triangleleft T_k = G$$

whose quotients T_i/T_{i-1} are rank one abelian, and such that T_i/T_{i-2} is not bi-orderable for any $i = 2, \ldots, k$. For all other groups, there are uncountably many left-orderings.

Bi-orderings behave differently. In this case, there are groups that admit finitely many, countably infinitely many, and uncountably many bi-orderings. However there are no "structure theorems" saying exactly which groups exhibit which kind of behaviour. I.e., the following is open:

Question 1.26. The following questions are open as of the time of writing.

1. Determine which groups admit finitely many bi-orderings.
2. Determine which groups admit countably infinitely many bi-orderings.
3. A nonidentity element $g \in G$ is *generalized torsion* if a product of conjugates of g is equal to the identity. If G is generalized torsion free, must G be LO? (Kourovka Notebook Problem 16.48)

2. Lecture 2: Quotients, direct products, free products, amalgams

This lecture investigates the behaviour of left and bi-orderability with respect to standard group-theoretic structures. We begin with two easy ones, direct products and quotients.

Proposition 2.1. *The direct product $G \times H$ is LO (resp. BO) if and only if G and H are both LO (resp. BO).*

Proof. Create a lexicographic ordering using short exact sequences, i.e. if $P_G \subset G$ and $P_H \subset H$ then define $(g, h) \in P$ if $g \in P_G$ or $g = id$ and $h \in P_H$. Then P is a positive cone in $G \times H$. □

More generally, if $\{G_i\}_{i \in I}$ is an arbitrary family of LO groups, then $\prod_{i \in I} G_i$ is LO, which we demonstrate by generalising the previous construction. Choose a well-ordering $\prec$ of the index set i, positive cones $P_i \subset G_i$ and then order elements of $\prod_{i \in I} G_i$ according to their first nonidentity entry (here, 'first' means with respect to the well-ordering $\prec$).

Definition 2.2. *A subgroup $H \subset G$ is convex with respect to the left-ordering $<$ of G, or $<$-convex for short, if for all $g, h \in H$ and $f \in G$, $g < f < h$ implies $f \in H$. A subgroup H of G is relatively convex if it is $<$-convex for some left-ordering $<$ of G.*

Proposition 2.3. *Suppose that G is LO and N is a normal subgroup. The quotient G/N is left-orderable if and only if N is relatively convex in G.*

Proof. Suppose that N is $<$-convex, and define a total ordering $\prec$ of G/N by $N \prec hN$ if and only if $1 < h$.

To see this is well-defined, suppose that $hN = h'N$ with $h' < id < h$. Then note that $id < (h')^{-1} < h^{-1}h$, and since $h^{-1}h \in N$ then $h' \in N$ by convexity, a contradiction.

Checking that this defines a left-ordering is straightforward from here. □

Proposition 2.4. *The free product $G * H$ is left-orderable if and only if G and H are both left-orderable. Moreover, if G admits a left-ordering $<_G$ and H admits a left-ordering $<_H$, then $G * H$ admits a left-ordering $<$ whose restriction to G is $<_G$, and whose restriction to H is $<_H$.*

Proof. We present two proofs.

Proof 1: Note there's a map $G * H \to G \times H$ that's induced by the maps $g \mapsto (g, id)$ and $h \mapsto (id, h)$. So there's a short exact sequence

$$\{id\} \to K \to G * H \xrightarrow{q} G \times H \to \{id\},$$

and we can analyze the kernel of this map as follows. Recall that every non-identity element of $G * H$ can be uniquely written as a product

$$w = a_1 a_2 a_3 \dots a_n$$

where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$, we call $\ell(g) = n$ the length of g and set $\ell(id) = 0$. We first note that K is generated by the set

$$S = \{[g, h] = ghg^{-1}h^{-1} \mid g \in G \setminus \{id\} \text{ and } h \in H \setminus \{id\}\}.$$

To see this, we induct on the length of $w \in K$, first noting that if $\ell(gw) = 0$ then w is trivially in $\langle S \rangle$. Now suppose that for $\ell(w) < n$ if $w \in K$ then $w \in \langle S \rangle$, and consider $w \in K$ with $\ell(w) = n$. First we can check that if $w \in K$ then $\ell(w)$ cannot be less than four, and then write

$$w = a_1 a_2 a_3 \dots a_n = (a_1 a_2 a_1^{-1} a_2^{-1})(a_2 a_1 a_3 a_4 \dots a_n) = [a_1, a_2]w'$$

where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$. Note that $\ell(w') < n$ since a_1 and a_3 are adjacent and come from the same factor, and so the induction assumption applies, landing w in $\langle S \rangle$.

Next, we can in fact check that S is a free basis, to do this we'll show that no reduced word in S represents id. Write $x_{g,h}$ in place of $[g,h]$ and suppose

$$w = x_{g_1,h_1}^{\epsilon_1} x_{g_2,h_2}^{\epsilon_2} \dots x_{g_n,h_n}^{\epsilon_n}$$

where $g_i \in G \setminus \{id\}$ and $h_i \in H \setminus \{id\}$, $\epsilon_i = \pm 1$ and you never have $(g_i, h_i) = (g_{i+1}, h_{i+1})$ and $\epsilon_i = -\epsilon_{i+1}$ for some $i \in \{1, \dots, n-1\}$. I.e, it's a reduced word in S.

We can prove that W can be written uniquely as an alternating product $a_1 a_2 a_3 \dots a_m$ where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$, and either $a_{m-1}a_m = g_n^{-1}h_n^{-1}$ if $\epsilon_n = 1$ or $a_{m-1}a_m = h_n g_n$ if $\epsilon_n = -1$. We prove this by inducting on the length of w in the generators $x_{g,h}$, the case of $n = 1$ being obvious.

Considering only the case $\epsilon_n = 1$ we first apply the induction assumption to $x_{g_1,h_1}^{\epsilon_1} x_{g_2,h_2}^{\epsilon_2} \dots x_{g_{n-1},h_{n-1}}^{\epsilon_{n-1}}$ to write w as either

$$w = a_1 a_2 \dots g_{n-1}^{-1} h_{n-1}^{-1} x_{g_n,h_n}^{\epsilon_n} = a_1 a_2 \dots g_{n-1}^{-1} h_{n-1}^{-1} g_n h_n g_n^{-1} h_n^{-1}$$

or

$$w = a_1 a_2 \dots h_{n-1} g_{n-1} g_n h_n g_n^{-1} h_n^{-1}.$$

Note that in the first case, we're done, and in the second case, $h_{n-1}g_{n-1}g_n h_n \neq id$ by our assumption that w is a reduced word in the $x_{g,h}$. So we're done in this case, too. The case of $\epsilon_n = -1$ is similar.

Now the result follows from

$$\{id\} \to K \to G * H \xrightarrow{q} G \times H \to \{id\},$$

as we can lexicographically order $G \times H$ using given orderings $<_G$ and $<_H$. Then K is free, so it's LO (in fact BO), and so we can lexicographically order $G * H$. Moreover, the ordering on $G * H$ extends $<_G$ and $<_H$. This concludes the first proof.

Proof 2: This is due to Dicks and Sunic [12]. Define a function $\tau : G * H \to \mathbb{Z}$ as follows. First, fix positive cones $P_G \subset G$ and $P_H \subset H$, and given an nonidentity element $w \in G * H \setminus \{id\}$ write it uniquely as

$$w = a_1 a_2 a_3 \dots a_n$$

where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$. Define $\eta(w)$ to be 0 if $\ell(w) = 1$, and otherwise:

$$\eta(w) = \begin{cases} 0 & \text{if } a_1, a_n \in G \text{ or } a_1, a_n \in H, \\ 1 & \text{if } a_1 \in G \text{ and } a_n \in H \\ -1 & \text{if } a_1 \in H \text{ and } a_n \in G \end{cases}$$

Recall that $w = a_1 a_2 a_3 \dots a_n$ and define

$$\tau(w) = |\{i \mid a_i \in P_G \cup P_H\}| - |\{i \mid a_i \in P_G^{-1} \cup P_H^{-1}\}| + \eta(w).$$

One can check that $\tau(w)$ is always odd, so that no $w \in G * H \setminus \{id\}$ satisfies $\tau(w) = 0$. Set

$$P = \{w \in G * H \setminus \{id\} \mid \tau(w) > 0\}.$$

Checking this is a positive cone is a quick case argument, and it obviously extends both P_H and P_G since $\tau(g) = +1$, $\tau(h) = +1$ whenever $g \in P_G$, $h \in P_H$. □

Remark 2.5. Both of the constructions in the previous proof can be generalized to the case of free products with arbitrarily many factors.

These constructions, while both useful in creating left-orderings of free products, don't necessarily create bi-orderings if the initial orderings on G and H are bi-orderings. In the case of the short exact sequence construction, we would need to find a bi-ordering of K that is invariant under conjugation (not always possible). In the case of the function τ, it's enough to note that if $g \in G \setminus \{id\}$ is positive and $h \in H \setminus \{id\}$ is negative then $\tau(gh) > 1$, while

$$\tau(g^{-1}(gh)g) = \tau(hg) = -1.$$

So τ never yields a bi-ordering. Nonetheless, we have:

Theorem 2.6 (Vinogradov [26])**.** *The free product $G * H$ is BO if and only if G and H are both BO. Moreover, if $<_G$ and $<_H$ are bi-orderings of the factors, then $G * H$ admits a bi-ordering that extending $<_G$ and $<_H$.*

This proof is rather involved, so we offer up a sketch of the proof only, due to Bergman [1].

Sketch. First, we lexicographically order $G \times H$ to create an ordering $<$. Next, set

$$\Gamma = \mathbb{Z}[G \times H],$$

the elements of this ring being finite formal sums $\sum_{i=1}^{m} r_i(g_i, h_i)$, $r_i \in \mathbb{Z}$, $g_i \in G$ and $h_i \in H$. This means the underlying abelian group of Γ is $\bigoplus_{(g,h)\in G\times H} \mathbb{Z}$, i.e. the direct sum of one copy of $\mathbb{Z}$ for each $(g, h) \in G \times H$. Order this sum, and thus Γ, reverse lexicographically using the ordering $<$ of the index set. In other words, we declare an element positive if the coefficient of its largest term (largest with respect to $<$) is a positive integer. Denote the resulting ordering of Γ by the same symbol, $<$.

Now consider $M_2(\Gamma[t])$, the ring of 2×2 matrices with entries from the polynomial ring $\Gamma[t]$. Each element of $M_2(\Gamma[t])$ can be written as a sum

$$\sum_{i=0}^{n} M_i t^i$$

where $M_i \in M_2(\Gamma)$. We extend $<$ to this ring as follows, denoting the resulting ordering by $<$ again:

Each 2×2 matrix has entries $a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}$. We'll order the positions of all matrices in the order they've just been written, i.e. $a_{1,1}$ is always considered the first position, $a_{1,2}$ the second, and so forth. Given $\sum_{i=1}^{n} M_i t^i$ as above, let n denote the least integer such that M_n is nonzero. Declare $\sum_{i=0}^{n} M_i t^i > 0$ if the first nonzero entry in M_n is positive with respect to the ordering $<$ of Γ.

E.g. suppose $G = \langle s \rangle$ and $H = \langle r \rangle$ are infinite cyclic groups, and that $G \times H$ is ordered according to $(s^n, r^m) > (0, 0)$ if and only if $n > 0$ or $n = 0$ and $m > 0$. Then consider this element of $M_2(\Gamma[t])$:

$$\begin{bmatrix} (3(s,r) + (s^2,r))t + (s^3,r)t^2 & (s^2,r) - 5(s^3,r^2) \\ (2(s,r) - 3(s,r^{-2}))t + (s^3,r)t^2 & 0 \end{bmatrix}$$

Write it as:

$$\begin{bmatrix} 0 & (s^2,r) - 5(s^3,r^2) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (3(s,r) + (s^2,r)) & 0 \\ (2(s,r) - 3(s,r^{-2})) & 0 \end{bmatrix} t + \begin{bmatrix} (s^3,r) & 0 \\ (s^3,r) & 0 \end{bmatrix} t^2$$

note that the first nonzero matrix is

$$\begin{bmatrix} 0 & (s^2,r) - 5(s^3,r^2) \\ 0 & 0 \end{bmatrix},$$

whose first nonzero entry is negative in the ordering of $\mathbb{Z}[G \times H]$ since the sign of $(s^2, r) - 5(s^3, r^2)$ is determined by the coefficient -5. So this element is negative in $M_2(\Gamma[t])$.

Finally, note that there's a homomorphism $\phi_G : G \to M_2(\Gamma[t])$ given by

$$\phi_G(g) = \begin{bmatrix} (g, id) & ((g, id) - (id, id))t \\ 0 & (id, id) \end{bmatrix},$$

and similarly a homomorphism $\phi_H : H \to M_2(\Gamma[t])$

$$\phi_H(h) = \begin{bmatrix} (id, id) & 0 \\ ((id, h) - (id, id))t & (id, h) \end{bmatrix}.$$

Together these give a homomorphism $\phi : G * H \to M_2(\Gamma[t])$ whose image lands in the group of units of $M_2(\Gamma[t])$. Similar to the Magnus expansion, we can check that this homomorphism is injective and that the ordering we've defined on $M_2(\Gamma[t])$ provides a bi-ordering of the image of ϕ.

We'll check the map is injective, as the argument is rather nice, whereas checking this defines a bi-ordering is a a bit of a slog through the definitions.

Given $w = g_1 h_1 \cdots g_n h_n \in G * H$ a reduced word with $g_i \in G \setminus \{id\}$, $h_i \in H \setminus \{id\}$, we can use a ping-pong type of argument to check $\phi(w)$ is not the identity. There are other cases, e.g. when w begins with a term from H and ends with a term from G, or both the first and last terms are from the same factor, but these cases are similar.

Partition the set of all column vectors of the form

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

where $A(t), B(t) \in \Gamma[t]$ into disjoint sets V_1, V_2, V_3, where V_1 is the set of such vectors with $\deg A(t) > \deg B(t)$, V_2 is the set with $\deg A(t) < \deg B(t)$ and V_3 is the set $\deg A(t) = \deg B(t)$. Now start with $\begin{bmatrix} (id, id) \\ (id, id) \end{bmatrix}$ and consider the product

$$\phi(g_1 h_1 \ldots g_n h_n) \begin{bmatrix} (id, id) \\ (id, id) \end{bmatrix}.$$

We compute

$$\phi(h_n) \begin{bmatrix} (id, id) \\ (id, id) \end{bmatrix} = \begin{bmatrix} (id, id) & 0 \\ ((id, h_n) - (id, id))t & (id, h_n) \end{bmatrix} \begin{bmatrix} (id, id) \\ (id, id) \end{bmatrix}$$

which is equal to

$$\begin{bmatrix} (id, id) \\ ((id, h_n) - (id, id))t + (id, h_n) \end{bmatrix}.$$

an element of V_2. But then left-multiplying by $\phi(g)$ for any $g \in G$ sends an element of V_2 into V_1, and then left multiplying any element of V_1 by $\phi(h)$ for $h \in H$ yields and element of V_2, etc. In any event, the product indicated above doesn't land back in V_3. Thus $\phi(g_1 h_1 \cdots g_n h_n)$ is not the identity. $\square$

The situation is not so easy when it comes to free products with amalgamation. Suppose that A, G, H are groups equipped with injective homomorphisms $\phi_1 : A \to G$, $\phi_2 : A \to H$, and let $S \subset G * H$ denote the set

$$S = \{\phi_1(a)\phi_2(a^{-1}) \mid a \in A\}.$$

The free product of G and H amalgamated along the ϕ_i's is the quotient group

$$G *_{\phi_i} H = G * H \big/ \langle\langle S \rangle\rangle.$$

This group is not always left-orderable, as the following example shows, while our experience with free products tells us that sometimes (e.g. for trivial amalgamations) it will certainly be a LO group.

Example 2.7. Set $K_i = \langle x_i, y_i \mid x_i y_i x_i^{-1} = y_i^{-1} \rangle$ for $i = 1, 2$. Let $A = \mathbb{Z} \oplus \mathbb{Z}$. Note that for each i, the subgroup $\langle y_i, x_i^2 \rangle$ is isomorphic to A. Define $\phi_1 : A \to K_1$ by $\phi(0, 1) = y_1$, and $\phi(1, 0) = x_1^2$ while $\phi_2 : A \to K_1$ is given by $\phi_2(0, 1) = x_2^2$, and $\phi_2(1, 0) = y_2$.

Next observe that both groups K_i are left-orderable since they fit into a short exact sequence with infinite cyclic kernel and quotient. Moreover, in every left-ordering of K_i with $id < y_i$ (there is at least one of these) we must have $y_i < x_i$ and therefore $y_i < x_i^2$. To see this, suppose not, say $x_i < y_i$. Then $y_i^{-1} x_i < id$, and since $x_i^{-1} < id$, so we also have $x_i^{-1} y_i^{-1} x_i < id$. But then $x_i^{-1} y_i^{-1} x_i = y_i < id$ because $x_i y_i x_i^{-1} = y_i^{-1}$, this is a contradiction.

Now considering the free product with amalgamation $K_1 *_{\phi_i} K_2$, suppose that it is left-orderable. Then the argument above applied to $K_1 \subset K_1 *_{\phi_i} K_2$ tells us that we must have $y_1 < x_1^2$ in every left-ordering of $K_1 *_{\phi_i} K_2$. On the other hand, $x_1^2 = \phi_1(1,0) = \phi_2(1,0) = y_2$ and $y_1 = \phi_1(0,1) = \phi_2(0,1) = x_2^2$, so this inequality forces $x_2^2 < y_2$, which is not possible. So $K_1 *_{\phi_i} K_2$ must not be left-orderable. □

However, we do know necessary and sufficient conditions. First, some notation. For a LO group G, set

$$\mathrm{LO}(G) = \{P \subset G \mid P \text{ is a positive cone }\}.$$

Note that $\mathrm{LO}(G)$ has an action by conjugation, because when P is a positive cone, so is gPg^{-1}. A family $N \subset \mathrm{LO}(G)$ is called *normal* if it is invariant under this G-action, i.e. $P \in N \Rightarrow gPg^{-1} \in N$ for all $g \in G$. There is a much more general statement of the following theorem that holds for general amalgams and fundamental groups of graphs of groups (due to Chiswell, [8]), but we will stick to the case of two factors.

Theorem 2.8 (Bludov-Glass [3])**.** *Suppose that A, G, H are groups equipped with injective homomorphisms $\phi_1 : A \to G$, $\phi_2 : A \to H$. The free product with amalgamation $G *_{\phi_i} H$ is left-orderable if and only if there exist normal families $N_1 \subset \mathrm{LO}(G)$ and $N_2 \subset \mathrm{LO}(G)$ satisfying*

$$(\forall P \in N_i)(\exists Q \in N_j) \text{ such that } \phi_i^{-1}(P) = \phi_j^{-1}(Q)$$

whenever $i, j \in \{1, 2\}$.

The proof is well beyond the scope of these notes. The basic idea is to use the normal families, together with some sophisticated set-theoretic constructions, to create a totally ordered set $(X, <)$ which admits an effective, order-preserving action by $G *_{\phi_i} H$.

Despite the rather technical conditions, this theorem already means that certain types of free products with amalgamation are always LO.

Corollary 2.9. *Suppose that G, H, A are as above, that G and H are LO and A is infinite cyclic. Then $G *_{\phi_i} H$ is LO.*

Proof. Just take $N_1 = \mathrm{LO}(G)$ and $N_2 = \mathrm{LO}(H)$. These are certainly normal, and if $P \in N_1$ then there are only two possibilities for $\phi_1^{-1}(P)$ since $\mathbb{Z} \cong A$ has only two left-orderings. Both of these possibilities arise as $\phi_2^{-1}(Q)$ for some $Q \in N_2$, because if $\phi_2^{-1}(Q)$ gives one of the positive cones, the $\phi_2^{-1}(Q^{-1})$ gives the other. This situation is entirely symmetric so the same argument shows every $\phi_2^{-1}(Q)$ for $Q \in N_2$ has a corresponding $P \in N_1$. □

More generally:

Theorem 2.10. *Suppose that G, H, A are as above, that G and H are LO and nilpotent. Then $G *_{\phi_i} H$ is left-orderable.*

Proof. We'll use one black box here, which is a result due to E. Formanek [13]:

If $P \subset G \setminus \{id\}$ is a semigroup and G is nilpotent, then there exists a positive cone $Q \subset G$ with $P \subset Q$.

So we can apply the same argument as in the previous proposition, setting $N_1 = \mathrm{LO}(G)$ and $N_2 = \mathrm{LO}(H)$ since $\mathrm{LO}(A) = \{\phi_i^{-1}(P) \mid P \in N_i\}$. □

In other situations, sometimes the obvious necessary condition turns out to be enough. What we mean here by "obvious necessary condition" is the following: If $G *_{\phi_i} H$ is LO and so contains a positive cone P, then clearly $P \cap G = P_G$ and $P \cap H = P_H$ are two positive cones that satisfy $\phi_1^{-1}(P_G) = \phi_2^{-1}(P_H)$. So the existence of "compatible cones" P_G and P_H, in the sense that they agree on the amalgamated subgroups, is always necessary. In fact, it is sometimes sufficient.

Theorem 2.11. *Suppose that G, H, A are as above, that $\phi_1(A)$ is central in G and $\phi_2(A)$ is central in H. Then $G *_{\phi_i} H$ is LO if and only if there exist $P_G \subset G$ and $P_H \subset H$ with $\phi_1^{-1}(P_G) = \phi_2^{-1}(P_H)$.*

Proof. The "only if" part holds in general. On the other hand, suppose that there exist P_G, P_H as in the statement of the theorem.

Set

$$N_1 = \{P \in \mathrm{LO}(G) \mid \phi_1^{-1}(P) = \phi_1^{-1}(P_G)\}$$

and set

$$N_2 = \{P \in \mathrm{LO}(H) \mid \phi_1^{-1}(P) = \phi_1^{-1}(P_H)\}.$$

To see that these families are normal, note that if $P \in \mathrm{LO}(G)$ then

$$gPg^{-1} \cap \phi_1(A) = gPg^{-1} \cap g\phi_1(A)g^{-1} = g(P \cap \phi_1(A))g^{-1} = P \cap \phi_1(A)$$

since $\phi_1(A)$ is central in G. Therefore if $P \in N_1$ then $gPg^{-1} \cap \phi_1(A) = P \cap \phi_1(A)$ and therefore $\phi_1^{-1}(gPg^{-1}) = \phi_1^{-1}(P) = \phi_1^{-1}(P_G)$. Similarly for N_2.

Moreover, by these same observations, every $P \in N_1$ satisfies $\phi_1^{-1}(P) = \phi_2^{-1}(P_H)$, and every $Q \in N_2$ satisfies $\phi_1^{-1}(P_G) = \phi_2^{-1}(Q)$. So these families satisfy the hypothesis of Theorem 2.8, and thus $G *_{\phi_i} H$ is LO. □

We can generalize the "if" direction of the last theorem to prove things like:

Theorem 2.12. *Suppose that G, H, A are as above. If there exist $P_G \subset G$ and $P_H \subset H$ that are positive cones of bi-orderings with $\phi_1^{-1}(P_G) = \phi_2^{-1}(P_H)$, then $G *_{\phi_i} H$ is LO.*

There is also another extremely significant class of examples that are expected to behave this same way, in the sense that it's good enough to match left-orderings of the factors on the amalgamating subgroups, and the normal families somehow "happen for free".

Conjecture 2.13. *Suppose that for $i = 1, 2$, the 3-manifold M_i is compact, connected, orientable and irreducible, with boundary $\partial M_i = T_i$ an incompressible torus. Fix a homeomorphism $\phi : T_1 \to T_2$ and set $M = M_1 \cup_\phi M_2$, whose fundamental group is $\pi_1(M) = \pi_1(M_1) *_{\phi_i} \pi_1(M_2)$ for some choice of injective homomorphisms $\phi_i : \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(M_i)$ determined by the gluing map ϕ (i.e. we want $\phi(\phi_1(a, b)) = \phi_2(a, b)$ for all $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$).*

Then $\pi_1(M)$ is LO if and only if there exist positive cones $P_1 \in \pi_1(M_1)$ and $P_2 \in \pi_1(M_2)$ such that $\phi_1^{-1}(P_1) = \phi_2^{-1}(P_2)$.

This is closely related to the L-space conjecture [5]. Moreover, the above conjecture can be generalized to multiple 3-manifold pieces, but it's simplest if we stick with two for the statement of our conjecture.

Thusfar we have said nothing about bi-orderability of free products with amalgamation.

Question 2.14. If G, H, A are as above, find necessary and sufficient conditions that guarantee the group $G *_{\phi_i} H$ is bi-orderable.

This question has been solved in several special cases, but to my knowledge it is quite wide open in general. E.g.

Theorem 2.15 (Bergman [1])**.** *Suppose that G_1, G_2 are two copies of the same group, $i : G_1 \to G_2$ is the identity map, and that $\phi_i : N \to G_i$ are inclusions whose image is normal in G_i satisfying $i \circ \phi_1 = \phi_2$. Then $G_1 *_{\phi_i} G_2$ is BO if and only if $\phi_i(N)$ is relatively convex for $i = 1, 2$.*

3. Lecture 3: Conradian orderings and local indicability

So far we have seen that there are three classes of groups with the following inclusions

$$\{\text{BO groups}\} \subset \{\text{LO groups}\} \subset \{\text{torsion free groups}\}$$

and we have seen that each inclusion is proper. This lecture investigates the structures that lie between LO and BO.

As a first attempt:

Definition 3.1. *A left-ordering $<$ of a group G is* Archimedean *if, whenever $g, h \in G$ are both positive there exists $n > 0$ such that $g^n > h$.*

However this turns out *not* to lie properly in between LO and BO:

Proposition 3.2. *Suppose that G admits an Archimedean left-ordering $<$. Then $<$ is a bi-ordering, and G is abelian.*

Proof. To see that $<$ is a bi-ordering, let P denote its positive cone. Suppose that $g \in G$ and $h \in P$.

Assuming $g > id$, we choose $n > 0$ such that $g < h^n$. Then $id < g^{-1}h^n$ and since g is positive, $id < g^{-1}h^n g$ so that $id < g^{-1}hg$ upon taking roots so that $g^{-1}hg \in P$.

If g is negative, suppose that $g^{-1}hg \notin P$. Then $id < g^{-1}h^{-1}g$ so that, by the previous paragraph, conjugation by g^{-1} will yield a positive element. I.e.

$$(g^{-1})^{-1}(g^{-1}h^{-1}g)g^{-1} = h^{-1} \in P,$$

a contradiction. Overall, $g^{-1}Pg \subset P$ for all $g \in G$, so the ordering is a bi-ordering.

Now there are two cases. If $g \in G$ is the least positive element in the ordering, then if $h \in G \setminus \langle g \rangle$, up to sign there's an $n > 0$ such that $g^n < h < g^{n+1}$, so that $id < g^{-n}h < g$, a contradiction. So in this case $G \cong \langle \mathbb{Z} \rangle$ and we're done.

Otherwise the ordering of G is dense, and we can assume g, h and $[g, h]$ are all positive. By density we can choose x with $id < x^2 < [g, h]$ (there's a little trick here to get x^2 instead of just x) and m, n with $x^m \leq g < x^{m+1}$ and $x^n \leq h < x^{n+1}$. But then multiplying inequalities (which we can do, since it's a bi-ordering) we arrive at $ghg^{-1}h^{-1} < x^{m+1}x^{n+1}x^{-m}x^{-n} = x^2$, a contradiction. ☐

Proposition 3.3. *An ordering $<$ of an abelian group A is Archimedean if and only if there are no $<$-convex subgroups in A.*

Proof. The key here is that in any abelian group, for any element $a \in A$ the set

$$D_a = \{g \in A \mid \exists n \in \mathbb{Z} \text{ such that } a^{-n} < g < a^n\}$$

is a convex subgroup. ☐

In fact we can completely characterize Archimedean ordered groups, finishing out investigation of this property.

Proposition 3.4 (Hölder's Theorem [16])**.** *Suppose that $(G, <)$ is an Archimedean ordered group. Then there exists an order-preserving injective homomorphism $\phi : G \to (\mathbb{R}, +)$, where $\mathbb{R}$ is equipped with the usual ordering.*

Proof. Choose a nonidentity element $f \in G$ and fix $\phi(f) = 1$. Now, by the Archimedean property, for each positive $g \in G$ and $n \in \mathbb{N}$ there exists a nonnegative integer a_n such that

$$f^{a_n - 1} \leq g^n < f^{a_n}.$$

Set

$$\phi(g) = \lim_{n \to \infty} \frac{a_n}{n}.$$

Note that since $<$ is a bi-ordering, given $m, n \in \mathbb{N}$ we can combine $f^{a_n-1} \leq g^n < f^{a_n}$ and $f^{a_m-1} \leq g^m < f^{a_m}$ to get

$$f^{a_n+a_m-2} \leq g^{n+m} < f^{a_n+a_m}$$

so that $a_{n+m} \leq a_n + a_m$, meaning $\{a_n\}$ is a subadditive sequence. Therefore by Fekete's lemma

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n}.$$

Since the a_n's are all nonnegative, this infimum is not $-\infty$, so we can set

$$\phi(g) = \inf \frac{a_n}{n}.$$

From here we need to check that this is both order-preserving, injective and a homomorphism–it follows from the definition of the a_n's. For example, if $g, h \in G$ then $f^{a_n-1} \leq g^n < f^{a_n}$ and $f^{b_n-1} \leq h^n < f^{b_n}$ implies

$$f^{a_n+b_n-2} \leq (gh)^n < f^{a_n+b_n}$$

so that if c_n is the unique integer with

$$f^{c_n-1} \leq (gh)^n < f^{c_n}$$

then $a_n + b_n - 1 \leq c_n \leq a_n + b_n$. Therefore

$$\phi(gh) = \lim_{n\to\infty} \frac{c_n}{n} = \lim_{n\to\infty} \frac{a_n + b_n}{n} = \phi(g) + \phi(h).$$

The other properties of ϕ are proved similarly. □

So Archimedean orders did not yield any generalization of LO or BO that was useful. Let us try something different, instead we will try orders that are built lexicographically from Archimedean pieces. Let us formalize this as follows.

Lemma 3.5. *Suppose that C, D are $<$-convex subgroups in the ordered group $(G, <)$. Then either $C \subset D$ or $D \subset C$ or $C = D$.*

Proof. The proof is routine. □

Definition 3.6. *A* convex jump *in an ordered group $(G, <)$ is a pair of $<$-convex subgroups (C, D) with $C \subset D$ such that if there is a third $<$-convex subgroup $C \subset C' \subset D$ then either $C = C'$ or $C' = D$.*

A convex jump (C, D) is Conradian *if C is normal in D and the quotient ordering of D/C is Archimedean. In this case, we call any homomorphism $\tau_{(C,D)} : D/C \to \mathbb{R}$ arising from Hölder's theorem the* Conrad homomorphism *associated to the jump (C, D).*

Example 3.7. Suppose that G fits into the short exact sequence

$$\{id\} \to \mathbb{Z}^2 \xrightarrow{i} G \xrightarrow{q} \mathbb{Z} \to \{id\}$$

and equip G with an ordering as follows.

Define an Archimedean ordering on $\mathbb{Z}^2$ with positive cone P by choosing $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$ with irrational slope, and declaring $(n_1, n_2) \in P$ if and only if $n_1v_1 + n_2v_2 > 0$. The usual ordering on $\mathbb{Z}$ is also Archimedean, say its positive cone is Q.

Set $R = i(P)+q^{-1}(Q)$, this is the positive cone of an ordering on G. The convex jumps associated to this ordering are $(\{id\}, i(\mathbb{Z}^2))$ and $(i(\mathbb{Z}^2), G)$, and the Conrad homomorphisms are

$$\tau_1 : \mathbb{Z}^2 \to (\mathbb{R}, +), \text{ where } \tau_1(\vec{n}) = proj_{\vec{v}}(\vec{n}).$$

and

$$\tau_2 : \mathbb{Z} \to (\mathbb{R}, +), \text{ where } \tau_1(n) = n.$$

□

Definition 3.8 (First appearing in [10], though with a different definition). *A left-ordering $<$ of a group G is* Conradian *if, for every $g \in G$ there exists a Conradian convex jump (C, D) such that $g \in D \setminus C$. A group that admits a Conradian left-ordering is called a Conradian LO group.*

At this moment, it's perhaps clear that we have defined some new kind of ordering, but its relationship with LO and BO is not clear at all. The next theorem will allow us to conclude that

$$\{ \text{ Conradian LO groups } \} \subset \{ \text{ LO groups } \}$$

is a proper containment. Recall that a group G is *locally indicable* if every finitely generated subgroup H of G admits a surjection $H \to \mathbb{Z}$.

Theorem 3.9. *If a group G admits a Conradian left-ordering, then it is locally indicable.*

Proof. First, suppose that G admits a Conradian left-ordering $<$. Given $H \leq G$ finitely generated, suppose the generators are $id < h_1 < h_2 < \cdots < h_k$. Then considering the jump (C, D) with $h_k \in D \setminus C$, we see that D must contain H since it contains all of the generators of H by convexity. On the other hand, C does not contain H. So there's a Conrad homomorphism $\tau : D/C \to (\mathbb{R}, +)$ which maps $H/(H \cap C)$ onto a nontrivial torsion-free abelian group. This gives a homomorphism from H onto a torsion-free finitely generated abelian group, so there's certainly a homomorphism from H onto $\mathbb{Z}$. □

Example 3.10. It is rather tricky to construct examples of left-orderable groups that are not locally indicable. Here is a family of examples that comes from 3-manifold theory, attributed by Bergman to Thurston and Kropholler [2].

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the circle S^1 by orientation-preserving homeomorphims. This action arises from identifying S^1 with the set of infinite rays in $\mathbb{R}^2$ emanating from the origin, and observing that $\mathrm{SL}(2, \mathbb{R})$ acts on the set of rays in a natural way. Moreover, since the determinant is every element in $\mathrm{SL}(2, \mathbb{R})$ is $+1$, the action is orientation-preserving, yielding an inclusion $\mathrm{SL}(2, \mathbb{R}) \to \widetilde{\mathrm{Homeo}}_+(S^1)$.

The universal cover of $\mathrm{SL}(2, \mathbb{R})$ is also a group, denoted $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and the action of $\mathrm{SL}(2, \mathbb{R})$ on the circle lifts to an effective action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ on the real line. Thus $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is left-orderable.

Consider the group

$$G = \langle x, y, z \mid x^2 = y^3 = z^7 = xyz \rangle,$$

there is an explicit embedding $G \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$ given in [2], meaning that G is left-orderable. However, upon abelianizing G we get $y^2 = xz$ from the relation $y^3 = xyz$. Then $x^2 = xyz$ yields $x = yz$, which combines with the previous relation to give $y^2 = (yz)z$ or $y = z^2$. Now $y^3 = z^7$ gives $z = id$, so that $y = id$ and $x = id$ as well. In the end, $G/G' = \{id\}$ so that there is no surjection $G \to \mathbb{Z}$. Thus G is LO, but not Conradian left-orderable. □

To strengthen this result and clarify the relationship with BO, we need an alternative characterization of Conradian orderability. This follows from the work of [10], but this proof is not stated there as below.

Theorem 3.11. *For a left-ordering $<$ of a group G, the following are equivalent:*

(1) *$<$ is a Conradian left-ordering.*
(2) *For every pair of positive elements $g, h \in G$, there exists $n > 0$ such that $g < hg^n$.*

Proof. We begin with a lemma:

Lemma 3.12. *If $\{C_i\}$ are $<$-convex subgroups of an ordered group $(G, <)$, then $\bigcup_{i \in I} C_i$ and $\bigcap_{i \in I} C_i$ are also $<$-convex subgroups.*

Proof. Once we know that $<$-convex subgroups are ordered by inclusion, the proof is just a matter of checking definitions. □

In light of this, if $<$ is a Conradian LO of G, then every convex jump has the following form: For each $g \in G$, set

$$C_g = \bigcup_{\substack{C \text{ convex} \\ g \notin C}} C \text{ and } D_g = \bigcap_{\substack{D \text{ convex} \\ g \in D}} D.$$

Then (C_g, D_g) is the Conradian jump associated to $g \in G$.

Now, let $g, h \in G$ be positive. We'll actually show that $g < hg^2$, to do this we need to show $id < g^{-1}hg^2$. To do this, consider the Conradian jumps (C_g, D_g) and (C_h, D_h) associated to g and h. Suppose that $D_g \subset C_h$.

Let $\tau : D_h/C_h \to (\mathbb{R}, +)$ be the Conrad homomorphism associated to the jump (C_h, D_h). Observe that $\tau(g^{-1}hg^2) = \tau(h)$ since $\tau(g) = 0$, and since h is positive and τ is order-preserving, $\tau(h) > 0$. This means $g^{-1}hg^2 > id$, again since τ is order preserving. Arguing similarly when $D_h \subset C_g$ and $(C_g, D_g) = (C_h, D_h)$ yields $g^{-1}hg^2 > id$ for all positive $g, h \in G$.

The difficult direction is long and technical, so we offer up a sketch. A complete proof appears in [9].

Step 1. Suppose G admits an ordering $<$ that satsifies (2) and (C, D) is a convex jump. Then if $g, h \in C \setminus D$ are positive elements, there exists $n > 0$ such that $g^n > h$.

To prove this, we fix $g > id$ and consider the set

$$\mathcal{X} = \{S \subset G \mid x \in S \text{ and } y < S \Rightarrow y \in S\}$$

ordered by inclusion, and observe that the G-action on $\mathcal{X}$ given by left-multiplication is order-preserving. Then consider the stabilizer of the element

$$S_0 = \{x \in G \mid x < g^n \text{ for some } n > 0 \ \}.$$

One can show that if $h \notin S_0$, then the stabilizer of S_0 is a convex subgroup lying properly between C and D, a contradiction to (C, D) being a convex jump.

Step 2. If G admits an ordering $<$ that satsifies (2) and (C, D) is a convex jump, then C is normal in D. Starting with $h \in D \setminus C$ and $h > id$, assume that $hCh^{-1} \not\subset C$. So we can choose $c \in C$ such that $hch^{-1} \notin C$ and WLOG assume $hch^{-1} > id$. Use Step 1 to find $n > 0$ such that $hc^nh^{-1} > h$, or $c^nh^{-1} > id$, meaning $h^{-1} > c^{-n}$. But then $id > h^{-1} > c^{-n}$ forces $h \in C$ by convexity, a contradiction. Dealing with $h < id$ is similar.

Step 3. Conclude. If (C, D) is a convex jump relative to an ordering satisfying (2), then by Step 2, C is normal in D. By Step 1, the quotient ordering of D/C is Archimedean. □

Remark 3.13. From the proof of the previous theorem, one can see that a left-ordering is Conradian if and only if whenever $g, h > id$ then $g < hg^2$. (I.e., $n = 2$ suffices).

Corollary 3.14. *Every bi-ordering is a Conradian ordering.*

Proof. If $g, h > id$ for some bi-ordering $<$ on a group G, then $id < h$ implies $g < hg$ simply by right-multiplying by g. The result now follows from Theorem 3.11. □

However it should be clear that not every BO group is a Conradian LO group, for instance $\langle x, y \mid xyx^{-1} = y^{-1} \rangle$ is not BO but is Conradian LO. This means we've done it, we have found something that lies (properly) in between LO groups and BO groups:

$$\{ \text{ BO groups } \} \subset \{ \text{ Conradian LO groups } \} \subset \{ \text{ LO groups } \}$$

with each containment proper. The utility of this construction might be in doubt, so let us push a little further to discover what "Conradian LO" actually means.

We can provide analogues of the LO and BO results in our first lecture, by mimicking the proofs found there.

Theorem 3.15. *Let $Q \subset G$ be a semigroup satisfying: if $g, h \in Q \setminus \{id\}$ then $g^{-1}hg^2 \in Q$. Then there exists a positive cone P of a Conradian ordering of G with $Q \setminus \{id\} \subset P$ if and only if Q satisfies $(*)$:*

For every finite set $\{g_1, \dots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin Csg(Q \setminus \{id\}, g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n})$.

Here, $Csg(S)$ is the smallest subsemigroup of G containing S which satisfies: if $g, h \in Csg(S) \setminus \{id\}$ then $g^{-1}hg^2 \in Csg(S)$ (the "Conradian" subsemigroup generated by S).

Proof. The proof goes through in an identical manner, though there is one change in the setup worth highlighting, namely we set

$$M = \{\text{Conradian semigroups } P \subset G \text{ with } Q \subset P \text{ that satisfy } (*)\}.$$

Then M is nonempty since it contains Q, it is partially ordered by inclusion as before, and one can check that every chain has an upper bound–but this is now a bit trickier to see! The key ingredient is a technical description of elements of $Csg(S)$, see [9, Lemma 9.18] for such a description. The proof concludes as before. $\square$

Corollary 3.16. *A group G is Conradian LO if and only if for every finite subset $\{g_1, \dots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin Csg(g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n})$.*

Proof. Take $Q = \{id\}$ in the previous theorem. $\square$

Corollary 3.17. *A group G is Conradian LO if and only if every finitely generated subgroup of G is Conradian LO.*

Now the surprising result is that in this context, the Burns-Hale theorem turns into an "if and only if" related to local indicability:

Theorem 3.18 (Conradian Burns-Hale, see e.g. [9])**.** *Every locally indicable group is Conradian left-orderable, and thus, a group is Conradian LO if and only if it is locally indicable.*

Proof. It is possible to simply repeat the Burns-Hale argument, see [9] for full details. The main technical ingredient we need is a concrete description of elements of $Csg(g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n})$, whenever $\{g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n}\} \subset G \setminus \{id\}$, as such w are no longer words in $\{g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n}\}$. However, one can prove the following lemma:

Lemma 3.19. *If $w \in Csg(g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n})$ then*

$$w = \prod_{i=1}^{k} g_{j_i}^{\epsilon_{j_i} m_i}$$

where $g_{j_i} \in \{g_1^{\epsilon_1}, \dots, g_n^{\epsilon_n}\}$ and the integers m_i satisfy, for each fixed $k \in \{1, \dots, n\}$,

$$\sum_{j_i = k} m_i > 0.$$

I.e., each generator $g_i^{\epsilon_i}$ occurs with positive exponent sum (c.f. [9, Lemma 9.18]*).*

Now proceed as in the previous proof of the Burns-Hale theorem. Namely, we first observe that $\langle g \rangle \cong \mathbb{Z}$ for any $g \in G \setminus \{id\}$, so choosing $\epsilon = \pm 1$ works to satisfy Corollary 3.16. Now induct, assuming that we can choose suitable exponents for all subsets of $G \setminus \{id\}$ of size $k-1$ or less.

Considering $\{g_1, \dots, g_k\} \subset G \setminus \{id\}$, choose a surjective homomorphism $\phi : \langle g_1, \dots, g_k \rangle \to \mathbb{Z}$ and assume that $\phi(g_i) = 0$ for $i = 1, \dots, r$ and $\phi(g_i) \neq 0$ for $i > r$, where $1 \leq r < k$. Choose exponents $\epsilon_i = \pm 1$ for $g_{r+1}, \dots, g_k$ so that $\phi(g_i^{\epsilon_i}) > 0$ for $i = r+1, \dots, k$ and choose exponents $\epsilon_i = \pm 1$ of $g_1, \dots, g_r$ using the induction assumption.

Now if $w \in Csg(g_1^{\epsilon_1}, \dots, g_k^{\epsilon_k})$ and w contains any occurrences of g_i for $i > r$, then $\phi(w) > 0$ by Lemma 3.19. Otherwise $w \in Csg(g_1^{\epsilon_1}, \dots, g_r^{\epsilon_r})$ and the induction assumption tells us $w \neq id$. So the result follows from Corollary 3.16. $\square$

So this establishes

$$\{\text{ BO groups }\} \subset \left\{ \begin{array}{c} \text{Conradian LO groups} \\ \text{i.e., locally indicable groups} \end{array} \right\} \subset \{\text{ LO groups }\}.$$

It happens that this can be further refined, using the notion of "recurrent orderings" introduced by Dave Morris [21]:

$$\{\text{BO groups}\} \subset \{\text{recurrent orderable groups}\} \subset \{\text{Conradian LO groups}\},$$

however we will not study recurrent orderings here, and refer the interested reader to [21]. However, it is worth noting that both containments above are proper, i.e. the class of recurrent orderable groups is different from the class of BO groups, and different from the class of locally indicable groups. As an example to show the first containment is proper, Dave Morris shows that if F is a free subgroup of finite index in $\mathrm{SL}(2, \mathbb{Z})$ then the semidirect product $F \rtimes \mathbb{Z}^2$ is Conradian LO, but admits no recurrent orderings.

Question 3.20 (Linnell [20])**.** Does every LO group which is not locally indicable contain a non-abelian free subgroup?

4. Lecture4: Locally invariant orderings and unique products

In this last lecture, we investigate what lies in between

$$\{\text{LO groups}\} \subset \{\text{torsion free groups}\}.$$

There have been many, many properties introduced over the years that lie between these two. Here is a list of all the ones I know about:

(1) Unique product property
(2) Two unique product property
(3) Weakly diffuse
(4) Diffuse
(5) Partially locally invariant orderable
(6) Totally locally invariant orderable.

Thankfully, it happens that some years after these concepts were introduced, (1) and (2) were shown to be equivalent by A. Strojnowski [25], and (3), (4), (5) and (6) were all shown to be equivalent by Dave Morris and Peter Linnell [19]. So the list is pared down considerably to something much simpler:

(1) Unique product property
(2) Totally locally invariant orderable

In this final lecture, we will flesh out this picture and discuss what's known, and what is not.

We introduce each of these notions as a weakening of left-orderability, where we insist on preserving a particularly desirable property. First, some motivation. One of the early drivers behind an investigation of LO groups was the Kaplansky zero-divisor conjecture (and the idempotent conjecture, and the units conjecture–though we know as of two years ago that the units conjecture is false):

Conjecture 4.1. *If G is torsion-free and K is a field, then $K[G]$ contains no zero divisors.*

Proposition 4.2. *If G is a LO group and K is a field, then $K[G]$ has no zero divisors.*

Proof. Suppose we have two nonzero elements of $K[G]$ and we take their product

$$\left(\sum_{i=1}^{n} r_i g_i\right)\left(\sum_{j=1}^{m} s_j h_j\right) = \sum_{i,j} r_i s_j g_i h_j$$

and suppose $h_1 < \cdots < h_m$. Consider the set $S = \{g_i h_j\}$ of all products where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Suppose gh is the largest element of S in the left-ordering.

Then since $g_i h_j < g_i h_m$ for all $j < m$ and for all g_i, we know $h = h_m$. Suppose $g = g_{i_0}$ for some $i \le i_0 \le n$, then if $g_i h_m = g_{i_0} h_m$ we get $g_i = g_{i_0}$. So out of all the products $g_i h_j$ in S, the maximal element occurs exactly once. Therefore the term corresponding to the maximal element in the sum $\sum_{i,j} r_i s_j g_i h_j$ is nonzero, so $K[G]$ has no zero divisors. □

Note that the essential element of the proof is that there exists elements g, h whose product gh is maximal with respect to the left-ordering, and thus it is unique and cannot cancel. Inspired by this, we make two definitions, each aiming to mimic a different interpretation of this special property:

Definition 4.3. . *Suppose that $A, B \subset G$ are finite nonempty subsets of a group G. If there exists $a \in A$ and $b \in B$ such that $a' \in A$ and $b' \in B$ with $ab = a'b'$ implies $a = a'$ and $b = b'$, then ab is called a* unique product *for the pair (A, B). A group G has the* unique product property *(we say G is UP for short) if for every pair of finite nonempty subsets $A, B \subset G$ there is a unique product for (A, B).*

Definition 4.4. *A* locally invariant ordering *of a group G is a partial ordering (i.e. transitive, irreflexive relation) $<$ of the elements of G such that for all $g, h \in G$ with $h \neq id$, either $gh > g$ or $gh^{-1} > g$. A group which admits a locally invariant ordering will be called a LIO group.*

Remark 4.5. We use partial orderings for increased flexibility in our arguments, though it turns out that if a group admits a partial LIO, then it admits a total LIO (Linnell-Morris, via a compactness type argument [19]). Moreover LO groups provide examples of UP and LIO groups.

Proposition 4.6 (Promislow (unpublished), Delzant [11], Chiswell [7])**.** *In a LIO group $(G, <)$, if S and T are two finite nonempty subsets then there is a maximal element in ST that is a unique product for (S, T).*

Proof. Let S, T be finite nonempty subsets and choose $g \in ST$ maximal. Suppose that g is not a unique product, so $g = st = s't'$ for $s, s' \in S$ and $t, t' \in T$ with $t' \neq t$.

Set $h = t^{-1}t' \neq id$. Then $gh = st' \in ST$ and $gh^{-1} = s't \in ST$, but since at least one of the inequalities $g < gh$ or $g < gh^{-1}$ holds, this is a contradiction to the maximality of g. □

Corollary 4.7. *LIO groups are UP.*

So we have arrived at

$$\{\text{LO groups}\} \subseteq \{\text{LIO groups}\} \subseteq \{\text{UP groups}\} \subseteq \{\text{TF groups}\}.$$

Example 4.8. (Promislow, technique due to Kionke and Raimbault [17]) Consider the crystallographic group generated by

$$a(x, y, z) = (x + 1, 1 - y, -z)$$
$$b(x, y, z) = (-x, y + 1, 1 - z)$$
$$c(x, y, z) = (1 - x, -y, z + 1).$$

One can verify that this group is isomorphic to the group

$$G = \langle a, b \mid a^2ba^2 = b, b^2ab^2 = a\rangle,$$

here we have eliminated c from the presentation owing to the fact that $abc = id$.
Claim: For sufficiently large $r > 0$, the set $S = \{g \in G \mid ||g(0, 0, 0)|| < r\}$ satisfies: for all $s \in S$, there exists $g \in G$ with $g \neq id$ such that $gs \in S$ and $g^{-1}s \in S$.

Note that the subgroup $\langle a^2, b^2, c^2\rangle$ is normal and isomorphic to $\mathbb{Z}^3$ generated by translations by even integers, and the quotient is $\langle a, b \mid a^2 = b^2 = (ab)^2 = id\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by the images of a and b. This will help visualize the coming sketch of an argument.

We will prove the claim by showing that for $r \gg 0$, if $||w|| < r$ then there exists $g \in G$ such that $||g^{-1}w|| < r$ and $||gw|| < r$. In order to avoid a messy general argument, we will only deal with a special case here, the case of $w = (4n_1, 4n_2, 4n_3)$ where n_1, n_2, n_3 are large integers. In this case, we choose $g = a(a^{-2n_1}b^{-2n_2}c^{-2n_3})$ (for a general w, we choose the even powers of a, b, c so that the point $t = (-2n_1, -2n_2, -2n_3)$ is as close as possible to the midpoint of the line segment connecting $-w$ to $(0, 0, 0)$).

Now we estimate:

$$||gw|| = ||(2n_1 + 1, 1 - 2n_2, -2n_3)|| \sim \frac{1}{2}||w|| < r$$

$$||g^{-1}w|| = ||(6n_1 + 1, 1 - 2n_2, -2n_3)|| < ||w|| < r.$$

Similar estimates work in general, with a choice of t as indicated, proving the claim.

Suppose G is LIO with locally invariant ordering $<$. Then (S, S) should have a unique product which is maximal with respect to $<$ for SS by Proposition 4.6, say it's $st \in SS$. Then we can choose $g \in G$ such that $g^{-1}s, gs \in S$. But then $g^{-1}st, gst \in SS$, and at least one of $g^{-1}st > st$, $gst > st$ holds, contradicting maximality. □

Remark 4.9. This argument can be improved in a couple ways.

(1) The argument above works in general for Bieberbach groups with finite holonomy group (i.e., torsion-free crystrallographic groups).
(2) The argument can also be improved to show that G above is actually NOT a UP group [24]. So the containment

$$\{\text{UP groups}\} \subseteq \{\text{TF groups}\}$$

is proper.

Clearly every left-ordering is a LIO ordering, but not all LIO orderings and left-orderings. Here is an easy example to see this.

Example 4.10. Suppose that A is a nontrivial subgroup of $\mathbb{Q}$. For each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\alpha > 0$, define a map $f_\alpha : \mathbb{Q} \to \mathbb{R}$ by

$$f_\alpha(r) = \begin{cases} r, & \text{if } r \geq 0 \\ -\alpha r & \text{if } r < 0. \end{cases}$$

Define a binary relation $\prec_\alpha$ of $A \subset \mathbb{Q}$ by $a \prec_\alpha b$ if and only if $f_\alpha(a) < f_\alpha(b)$. Since the usual ordering on $\mathbb{R}$ is a strict total order and f_α is injective, $\prec_\alpha$ is a strict total order on A.

It's also not a left-ordering. Suppose that $0 < a < b$ for some $a, b \in A$. Then $a \prec_\alpha b$, however since $a - b < 0$ then $0 < -\alpha(a - b)$, so that $0 \prec_\alpha a - b$. In other words, subtracting b from both sides of $a \prec_\alpha b$ flips the inequality $\prec_\alpha$.

However $\prec_\alpha$ is a locally invariant total ordering. Let $a, b \in A$ such that $b \neq 0$, we'll show that either $a \prec_\alpha a + b$ or $a \prec_\alpha a - b$. Upon replacing b by $-b$, it suffices to check when $0 < b$. If $0 \leq a$, then $f_\alpha(a) = a < a + b = f_\alpha(a + b)$ and so $a \prec_\alpha a + b$. If $a < 0$, then $f_\alpha(a) = -\alpha a$ and $f_\alpha(a - b) = -\alpha(a - b)$. Since $a - b < a$, we have $-\alpha a < -\alpha(a - b)$ and so $a \prec_\alpha a - b$. Therefore, $\prec_\alpha$ is a locally invariant total ordering on A. □

With respect to standard constructions, the property of being an LIO group behaves the same as the property of being an LO group.

Proposition 4.11 ([7], [4]). *Suppose that*

$$\{id\} \to K \xrightarrow{i} G \xrightarrow{q} H \to \{id\}$$

is a short exact sequence of groups. If K and H are LIO groups, then so is G.

Proof. Choose a complete set of coset representatives T for $i(K)$ in G. If $<_K$ and $<_H$ are locally invariant orderings on K and H, define $<_G$ on G according to the rule $g <_G h$ if and only if $q(g) <_H q(h)$ or $q(g) = q(h)$ and $t^{-1}g <_K t^{-1}h$. Here, $t \in T$ is the coset representative chosen so that $g(i(K)) = t(i(K)) = h(i(K))$.

To see this works, suppose that $g, h \in H$ with $h \neq id$. Then if $q(gh) \neq q(g)$ or $q(gh^{-1}) \neq q(g)$ then $q(h) \neq id$ so that $<_G$ satisfies $gh >_G g$ or $gh^{-1} >_G g$ since $<_H$ is a locally invariant ordering.

On the other hand if $q(gh) = q(g)$ and $q(gh^{-1}) = q(g)$ then let t denote the coset representative of $g(i(K))$. In this case, we need to show that either $t^{-1}g <_K t^{-1}gh$ or $t^{-1}g <_K t^{-1}gh^{-1}$, but this follows immediately since $<_K$ is a locally invariant ordering. □

Corollary 4.12. *The free product of LIO groups is LIO.*

Proof. Free groups are LIO since the are BO. For LIO groups H, G, use the short exact sequence

$$\{id\} \to K \to G * H \xrightarrow{q} G \times H \to \{id\},$$

where we have already seen that K is free. □

So LIO groups behave as LO groups with respect to the usual constructions of direct and free products. In the case of quotients, there is a similar notion of convex subgroup, and the case of amalgams is only partly understood. In any event, to date every construction that allows you to create new LIO groups from old LIO groups also creates new LO groups from old LO groups. So, if you start with two LIO groups that are also LO and apply any common group-theoretic construction, you'll still get an LIO group which is also LO. This means that none of these methods can be used to construct an example that shows

$$\{\text{ LO groups }\} \subset \{\text{ LIO groups }\}$$

is a proper containment.

We therefore develop a new method for finding LIO groups, and this method uses a connection to δ-hyperbolic metric spaces. We begin with an obvious lemma.

Lemma 4.13. *A group G is LIO if and only if there exists a partially ordered set $(S, <)$ and a function $\phi : G \to S$ such that for all $g, h \in H$ with $h \neq id$, either $\phi(gh) > \phi(g)$ or $\phi(gh^{-1}) > \phi(g)$.*

Proof. Define a locally invariant ordering $<$ on G by $g < h$ if and only if $\phi(g) < \phi(h)$. □

Recall that a *geodesic segment* from x to y in a metric space (X, d) is an isometric embedding of an interval $[a, b] \subset \mathbb{R}$, $\alpha : [a, b] \to X$ satisfying $\alpha(a) = x$ and $\alpha(b) = y$. A geodesic metric space is a metric space (X, d) such that for all $x, y \in X$ there is a geodesic segment from x to y.

The *Gromov product* on X is defined as

$$\langle x, y\rangle_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y)),$$

and a space X is δ-hyperbolic if

$$\langle x, y\rangle_z \geq \min\{\langle x, t\rangle_z, \langle t, y\rangle_z\} - \delta$$

for all $t, x, y, z \in X$.

Given three points $x, y, z \in X$, a *triangle* in X with vertices x, y, z is a union of three segments joining x to y, y to z, and z to x. Denote these segments by $[x, y]$, $[y, z]$ and $[z, x]$. We say that a triangle is δ-thin at the vertex x if for all $p \in [x, y]$ and $q \in [x, z]$ satisfying $d(x, p) = d(x, q) \leq \langle y, z\rangle_x$, we have $d(p, q) \leq \delta$, see Figure 2; a triangle is δ-thin if it is δ-thin at all its vertices.

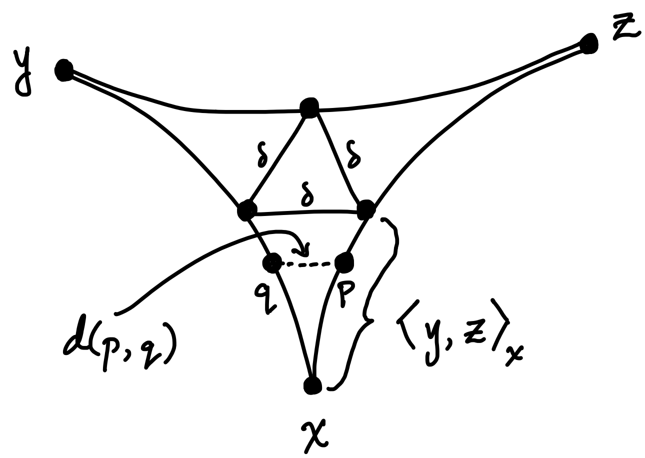

FIGURE 2. A δ-thin triangle.

Lemma 4.14 (Following [15]). *In a δ-hyperbolic space, all triangles are 4δ-thin.*

Proof. Suppose we have a triangle with edges $[x, y]$, $[y, z]$ and $[z, x]$, and that $p \in [x, y]$ and $q \in [x, z]$ with $d(x, p) = d(x, q) \leq \langle y, z \rangle_x$. Set $t = d(x, p)$, then one checks that

$$t = \langle p, y \rangle_x = \langle q, z \rangle_x.$$

Then by δ-hyperbolicity

$$\langle p, q \rangle_x \geq \min\{\langle p, y \rangle_x, \langle y, q \rangle_x\} - \delta$$

and

$$\langle y, q \rangle_x \geq \min\{\langle y, z \rangle_x, \langle z, q \rangle_x\} - \delta$$

so overall

$$\langle p, q \rangle_x \geq \min\{\langle p, y \rangle_x, \langle y, z \rangle_x, \langle z, q \rangle_x\} - 2\delta = t - 2\delta.$$

But $\langle p, q \rangle_x = t - \frac{1}{2}d(p, q)$, so that

$$t - \frac{1}{2}d(p, q) \geq t - 2\delta \Rightarrow d(p, q) \leq 4\delta.$$

$\square$

Lemma 4.15 (Delzant [11]). *Suppose that (X, d) is a geodesic Gromov δ-hyperbolic metric space. Suppose that h is an isometry of X such that $d(x, hx) > 6\delta$ for all $x \in X$. Then for every isometry g of X and all $x \in X$, either $d(x, ghx) > d(x, gx)$ or $d(x, gh^{-1}x) > d(x, gx)$.*

Proof. We prove this by contradiction. So let h be an isometry with $d(x, hx) > 6\delta$ for all $x \in X$, g another isometry of X, and assume that there is a point $p \in X$ such that

$$d(p, ghp) \leq d(p, gp) \text{ and } d(p, gh^{-1}p) \leq d(p, gp).$$

First, note that

$$\langle p, ghp \rangle_{gp} = \frac{1}{2}(d(p, gp) + d(ghp, gp) - d(p, ghp)),$$

and since $0 \leq d(p, gp) - d(p, ghp)$, we arrive at

$$\langle p, ghp \rangle_{gp} \geq \frac{1}{2}(d(ghp, gp)).$$

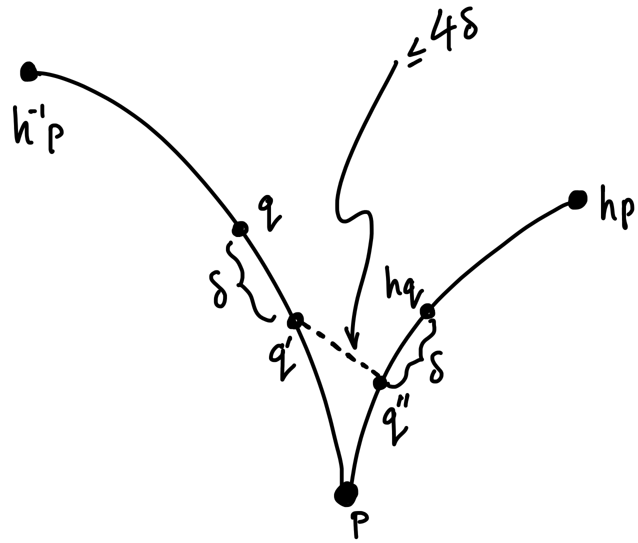

FIGURE 3. The segments σ and $h\sigma$, together with the points q, hq, q' and q''.

This fact is invariant under isometry, so we can apply g^{-1} to everything and arrive at

$$\langle g^{-1}p, hp\rangle_p \geq \frac{1}{2}(d(hp,p)).$$

On the other hand,

$$\langle p, gh^{-1}p\rangle_{gp} = \frac{1}{2}(d(p,gp) + d(gh^{-1}p, gp) - d(p, gh^{-1}p)),$$

and then using $0 \leq d(p,gp) - d(p,gh^{-1}p)$ we get

$$\langle p, gh^{-1}p\rangle_{gp} \geq \frac{1}{2}(d(gh^{-1}p, gp)),$$

or equivalently, since g and h are isometries

$$\langle g^{-1}p, h^{-1}p\rangle_p \geq \frac{1}{2}(d(h^{-1}p,p)) = \frac{1}{2}(d(hp,p)).$$

Using the fact that the space is δ-hyperbolic, we get

$$\langle h^{-1}p, hp\rangle_p \geq \min\{\langle h^{-1}p, gp\rangle_p, \langle gp, hp\rangle_p\} - \delta$$

which upon applying our prepared inequalities yields

$$\langle h^{-1}p, hp\rangle_p \geq \frac{1}{2}(d(hp,p)) - \delta.$$

Now consider the geodesic segment σ joining p to $h^{-1}p$, and let q denote its midpoint. Then hq is the midpoint of $h\sigma$ which joins hp to p. See Figure 3.

By assumption, the distance from hq to q is greater than 6δ. Let q' denote the point on the segment $[p,q]$ that is a distance δ from q, and q'' the point on the segment $[hq,p]$ that is a distance δ from hq. The distances $d(p,q')$ and $d(p,q'')$ are both equal to $\frac{1}{2}(d(hp,p)) - \delta$, so $\langle h^{-1}p, hp\rangle_p$ is greater than both of them.

This means we have arrived at the situation in Figure 3 and that we can use Lemma 4.14 and conclude that $d(q',q'') < 4\delta$. But then the triangle inequality gives

$$d(q,hq) \leq d(q,q') + d(q',q'') + d(hq,q'') < 6\delta,$$

a contradiction. □

Proposition 4.16 (Chiswell [7])**.** *Suppose that (X, d) is a geodesic Gromov δ-hyperbolic metric space and that G acts on X by isometries, and that for all $h \in G$ with $h \neq id$ and for all $x \in X$, we have $d(hx, x) > 6\delta$. Then G is LIO.*

Proof. Fix $p \in X$ and define a map $\phi_p : G \to \mathbb{R}$ by $\phi_p(g) = d(gp, p)$. Then by the previous lemma, for all $g \in G$ either $\phi_p(gh) > \phi_p(g)$ or $\phi_p(gh^{-1}) > \phi_p(g)$, making G a LIO group. □

Remark 4.17. Note that this proof requires that the segment joining p to $h^{-1}p$ has a midpoint– and if you choose an "exotic" metric space where there are no midpoints for segments, this proof won't necessarily apply. E.g. if you choose a metric space with a metric taking values in $\mathbb{Z}$, then a segment is an isometric embedding of $[n, m]$ for some $n < m$ in $\mathbb{Z}$. If p, $h^{-1}p$ are one apart, there's no midpoint and the proof fails. We can fix this by re-doing the proof for Λ-metric spaces (where Λ is an arbitrary ordered abelian group) and add the requirement that $d(p, hp) \in 2\Lambda$ for all $h \in G$ in order to get that G is LIO.

Maybe the requirement that every element of the group "moves points sufficiently far" may seem uncommon. In fact it is quite common.

Recall that a group G is *word-hyperbolic* if it is finitely generated, and if the geometric realization of a Cayley graph of G with respect to some (and therefore all) generating sets is a δ-hyperbolic metric space with respect to the metric induced by the word metric with unit edge length.

A group G is *residually finite* if, given $h \in G$ with $h \neq id$, there exists a finite-index normal subgroup N of G such that $h \notin N$. Equivalently, for every finite collection $\{h_1, \ldots, h_n\} \subset G$ there exists a finite-index normal subgroup N of G such that $h_i \notin N$ for all i. To see that one implies the other, for each h_i choose N_i finite index and normal in G with $h_i \notin N_i$, then set $N = \bigcap_{i=1}^{n} N_i$.

Proposition 4.18 (Chiswell [7])**.** *Suppose that G is a residually finite word hyperbolic group. Then G has a finite index LIO subgroup.*

Proof. Since a word hyperbolic group is finitely generated, we choose a finite generating set S and let Γ denote the geometric realization of the Cayley graph of G with respect to S (here we work with the geometric realization to avoid certain technicalities). This geometric realization is a δ-hyperbolic metric space for some $\delta > 0$.

Let
$$B = \{g \in G \mid d(id, g) \geq 6\delta\},$$
where d is the path metric on Γ with unit edge length, and we are identifying the vertices of with elements of G. Since the Cayley graph of G is locally finite, B is finite.

Since G is residually finite, it's fully residually finite, so there is a finite-index normal subgroup N of G with $B \cap N = \{id\}$. Now let $h \in N$ with $h \neq id$ and suppose there exists $x \in \Gamma$ with $d(hx, x) \leq 6\delta$. Then WLOG we can assume that x is a vertex, corresponding to an element of G, say $g \in G$. Then $d(hg, g) \leq 6\delta$ implies that $d(g^{-1}hg, id) \leq 6\delta$ so that $g^{-1}hg \in B$, a contradiction since $g^{-1}hg \in N$. □

Corollary 4.19 (Chiswell [7])**.** *If G is the fundamental group of a compact hyperbolic manifold, then G is virtually LIO (i.e. has a LIO subgroup of finite index).*

Proof. This follows from the fact that fundamental groups of compact hyperbolic manifolds are linear and finitely generated, hence residually finite by Malcev's Theorem (or Selberg's Lemma). □

However we don't have to pass to finite index subgroups in some cases. One can show that $\mathbb{H}^n$, hyperbolic n-space, is $\log(2)$-hyperbolic (see [23]) in the sense of Gromov. If M is a complete hyperbolic manifold, then $\pi_1(M)$ acts on the universal cover $\mathbb{H}^n$ by deck transformations. If we can somehow guarantee that the action satisfies $d(\gamma x, x) > 6\log(2)$ for all $x \in \mathbb{H}^n$, then this will guarantee that $\pi_1(M)$ is LIO. The notion we need is *injectivity radius.* And in fact, we can improve the required distance from $6\log(2)$ to $\log(1+\sqrt{2})$.

Theorem 4.20 (Bowditch [4])**.** *Suppose that M is a complete hyperbolic manifold of injectivity radius greater than $\log(1+\sqrt{2})$. Then $\pi_1(M)$ is LIO.*

Remark 4.21. This result of Bowditch is written in the language of diffuse groups, which was later shown to be equivalent to LIO [19]. Bowditch's result is stronger than that in these notes, in the sense that his result requires a smaller injectivity radius—this arises from the fact that his proof takes a different approach than ours.

Theorem 4.22 (Dunfield [17])**.** *There exists a 3-manifold M with $\pi_1(M)$ not LO and injectivity radius greater than $\log(1+\sqrt{2})$. Therefore the containment*

$$\{\ \textit{LO groups}\ \} \subset \{\ \textit{LIO groups}\ \}$$

is proper.

This also naturally raises the question:

Question 4.23. What property must a LIO group have in order to be a LO group? I.e. is there a property of groups, let us call is "property A," such that a group G is LO if and only G is LIO and has property A?

Question 4.24. Is the containment

$$\{\ \text{LIO groups}\ \} \subset \{\ \text{UP groups}\ \}$$

proper? I.e. does there exists a UP, non-LIO group?

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