Lie's third theorem

Def: A Lie group G is a group that is also a smooth manifold such that the (algebraic) maps $\mu: G\times G \to G$ $(g,h) \mapsto gh$, and $\iota: G \to G$, $g \mapsto g^{-1}$, are smooth. (Can use inverse function thm to show inverse is automatically smooth.)

Ex: $(V,+)$, $\mathbb{R}\setminus\{0\}$, $\mathbb{C}\setminus\{0\}$, GL_n (+ other matrix groups)
$S^1 \subset \mathbb{C}$, $S^3 \cong SU(2)$ (S^0, S^1, S^3 only spheres that are Lie groups)

Def: A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear binary operation denoted $[\,,\,]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto [X,Y]$, that satisfies

① $[X,Y] = -[Y,X]$

② $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$

Ex: $\mathfrak{X}(M)$, $(V, [\,,\,]=0)$, $(\mathbb{R}^3, \times)$, $(\operatorname{End} V, [\,,\,]\text{ comm.})$
(and as we'll soon see T_eG, where G is Lie grp)
e = identity elt.

Goal:

$$\text{Lie groups} \xrightarrow{\text{tg sp.}} \text{Lie algebras } (\dim < \infty)$$
$$\twoheadrightarrow$$
(Lie's third thm)

Def: Let G be a Lie group. For any $g \in G$, let $L_g : G \to G$ be $L_g(h) = gh$.
A vector field $X \in \mathfrak{X}(G)$ is <u>left-invariant</u> if $(L_g)_* X = X \quad \forall g \in G$.

Recall pushforward: F diffeo.

$$F^{-1}(q),\ X \xrightarrow{\quad} q,\ (F_* X)_q = (T_{F^{-1}(q)} F)(X_{F^{-1}(q)})$$

So for X left-inv., we have

$$h,\ X_h \xrightarrow{L_g} gh,\ X_{gh} = ((L_g)_* X)_{gh} = (T_h L_g)(X_h)$$

In particular, taking $h = e$, we see $X_g = (T_e L_g)(X_e)$ is determined by $X_e \in T_e G$; and any $v \in T_e G$ determines a left-inv. v.f. v^L, $\quad v^L_g = (T_e L_g)(v)$.

Denote $\{\text{left-invariant v.f.'s}\} = \mathfrak{X}(G)^L$

NB $\Rightarrow \dim \mathfrak{X}(G)^L < \infty$.

Prop: If $X, Y \in \mathfrak{X}(G)^L$, then $[X,Y] \in \mathfrak{X}(G)^L$.

proof: $X \sim_{L_g} X$, $Y \sim_{L_g} Y \Rightarrow [X,Y] \sim_{L_g} [X,Y] \iff [X,Y] \in \mathfrak{X}(G)^L$. ∎

Cor: $[u,v] := [u^L, v^L]_e$ defines a Lie bracket on $\mathfrak{g} = T_e G$.

Ex ① $G = (\mathbb{R}, +)$, $\mathfrak{g} = T_0\mathbb{R} \cong \mathbb{R}$, $[\,,\,] = 0$ (dim = 1); similarly for $G = S^1$.

② $G = GL_n(\mathbb{R}) \subset \mathrm{Mat}_{n\times n} = \mathbb{R}^{n^2}$ (open subset)

$\Rightarrow \mathfrak{g} = T_I G \cong \mathrm{Mat}_{n\times n} =: \mathfrak{gl}_n$

Lg linear

Bracket? Take $A \in \mathfrak{gl}_n$. $(A^L)_g = T_I L_g(A) = gA$.

coord of matrix

To unpack this, take (global) coordinates x_{ij} on G,

so basis for tangent space is $\left\{ \frac{\partial}{\partial x_{ij}}\Big|_g \right\}$

Can grind out formula in coord's to get

$$[A^L, B^L](g) = g(AB - BA), \text{ so } [\,,\,] = \text{commutator}.$$

Prop Let $\phi: G \to H$ be a Lie group homomorphism. Then $\phi_* = T_e\phi = d\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

proof: It suffices to show that for any $v \in \mathfrak{g}$, $(\phi_* v)^L \sim_\phi v^L$, for then given $u, v \in \mathfrak{g}$,

$$[(\phi_* u)^L, (\phi_* v)^L] \sim_\phi [u^L, v^L] \Rightarrow \phi_*([u,v]) = (T_e\phi)([u^L, v^L]_e) = [(\phi_* u)^L, (\phi_* v)^L]_e = [\phi_* u, \phi_* v].$$

Since $\phi(gh) = \phi(g)\phi(h)$, we have $\phi \circ L_g = L_{\phi(g)} \circ \phi$. Therefore,

$$(T_g\phi)(v^L_g) = (T_g\phi)(T_e L_g)(v) = T_e(\phi \circ L_g)(v) = T_e(L_{\phi(g)} \circ \phi)(v) = (T_e L_{\phi(g)})(T_e\phi)(v) = T_e L_{\phi(g)}(\phi_* v) = (\phi_* v)^L_{\phi(g)}$$ ∎

(So, we have a functor LieGrps $\longrightarrow$ Lie Alg.)

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we may define the following distribution D on G, $g \mapsto D_g = \{ v^L_g = (T_e L_g)(v) \mid v \in \mathfrak{h} \} \subset T_g G$.

The distribution D is involutive: Indeed, let $\{v_1, \ldots, v_r\}$ be a basis for $\mathfrak{h}$. Then the vector fields $v_1^L, \ldots, v_r^L$ span D. Let $X, Y \in \mathfrak{X}(G)$ that lie in D, and write $X = \sum_i a_i v_i^L$, $Y = \sum_j b_j v_j^L$.

Exercise: $[X,Y] = \sum_{i,j} a_i a_j [v_i^L, v_j^L] + a_i v_i^L(b_j) v_j^L - b_j v_j^L(a_i) v_i^L$.

$\Rightarrow [X,Y]$ lies in D.

By Frobenius theorem, $\exists$ maximal connected integral submanifold H through $e \in G$. Observe that for $g \in H$, $L_{g^{-1}}(H) \subset G$ is also an integral submanifold through e. Why? The diffeo. $L_{g^{-1}}$ preserves D: $(T_h L_{g^{-1}})(v_h^L) = (T_h L_{g^{-1}})(T_e L_h)(v) = (T_e L_{g^{-1}h})(v) = v^L_{g^{-1}h}$, and thus if $v \in \mathfrak{h}$, then $T_h L_{g^{-1}}$ sends $v_h^L \in D_h$ to $v^L_{g^{-1}h} \in D_{g^{-1}h}$. Therefore, $L_{g^{-1}}(H)$ is also an integral submanifold and it contains $L_{g^{-1}}(g) = g^{-1}g = e$. In other words, $g \in H, h \in H \Rightarrow g^{-1}h \in H$, or H is a subgroup.

This proves most of the following

Thm Let G be a Lie group w/ Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there exists a unique connected Lie subgroup H whose Lie algebra is $\mathfrak{h}$.

pf (sketch) Argument above shows we have an immersed submanifold H that is also a subgroup. It also shows $T_e H = \mathfrak{h}$.

It remains to show H is a Lie subgroup, and that H is a unique such Lie subgroup. To show $\mu: H \times H \to H$, $(g,h) \mapsto g^{-1}h$ is smooth: it's clear that $H \times H \to G$ is smooth. Recall H is a leaf of a foliation that "integrates" the distribution D above. Since smoothness is a local property, we may find nbhds $\mathcal{U}$ around (g,h) and V around $g^{-1}h$ s.t.

as map $H \times H \to G$ $\ni \mu(\mathcal{U}) \subset V$ and V is a submanifold chart for H, showing μ (as map $H \times H \to H$) is smooth.

Uniqueness — proof/sketch omitted. ∎

Finally, we sketch how one can associate to any finite dimensional Lie algebra $\mathfrak{g}$ a Lie group G with that Lie algebra.

<u>Thm</u> (Ado's theorem) Let $\mathfrak{g}$ be a finite dim. Lie algebra. Then $\exists\, n>0$ and an injective Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(n)$.

Therefore, we may view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}(n) = \mathrm{Lie}(GL(n))$. By a prior theorem, $\exists$ a Lie subgroup $G \subset GL(n)$ whose Lie algebra is $\mathfrak{g}$. ← Lie's third thm.

Remark: Lie subgroups of $GL(n)$ are called "matrix Lie groups". So every fin. dim. $\mathfrak{g}$ is the Lie algebra of a matrix Lie group. However, there <u>are</u> Lie groups G which are not isomorphic to any matrix Lie group (compare Ado's theorem).

Finally, consider now the following: does every Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{h}$ arise as the tangent map $\varphi_* = T_e\varphi$ of a Lie group homom. $\varphi: G \longrightarrow H$?

Similar to discussion above, the topology comes into play:

e.g. $\mathrm{id}: \mathbb{R} \to \mathbb{R}$ is a Lie algebra homomorphism. And $\mathbb{R} = \mathrm{Lie}(S^1)$ and $\mathbb{R} = \mathrm{Lie}(\mathbb{R})$. But $\nexists$ any non-trivial Lie group homomorphisms $S^1 \longrightarrow \mathbb{R}$. (Why?)

(But $\mathbb{R} \longrightarrow S^1$, $t \mapsto e^{it}$ is a homomorphism whose derivative at 0 is $\mathrm{id}_{\mathbb{R}}$.)

<u>Thm</u>: Let G, H be Lie groups w/ Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. If G is simply connected, then every Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{h}$ is the tangent map of a unique Lie group homomorphism.

proof — omitted.