A Step towards the Character Variety & the A-polynomial of a knot

(based on lecture notes by Boyer)

Thm (Waldhausen, Mostow, Scott, Perelman)
W_1, W_2 closed, connected, orientable, irreducible, 3-mfds not Lens spaces,
Then $\pi_1(W_1) \cong \pi_1(W_2) \iff W_1 \simeq W_2$

[Also extends to 3-mfds with bdy, modulo some details]

Study 3-mfds $\longleftrightarrow$ Study $\pi_1(M)$
$\uparrow$
representations into $SL(2,\mathbb{C})$

Rmk: A complete finite volumn (orientable) hyperbolic 3-mfd admits an essentially unique holonomy representation $\rho_M : \pi_1(M) \to PSL(2,\mathbb{C})$ which can be lifted to $SL(2,\mathbb{C})$

$SL(2,\mathbb{C})$: simple enough to give rich results.

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π: f.g group

$\pi = \langle \gamma_1, \dots, \gamma_n \mid r_i, i \in I \rangle$

[$\pi = \pi_1(M)$ 3-mfd group]

$R(\pi) = \{ \rho : \pi \to SL(2,\mathbb{C}) : \rho$ is a homomorphism $\}$

potentially encodes deep properties of π (hence M)

π: discrete top.

$SL(2,\mathbb{C})$: subspace top $\subset \mathbb{C}^4$

$(SL(2,\mathbb{C}))^n$: subspace top $\subset \mathbb{C}^{4n}$

$R(\pi)$: cpt-open top

Then:

$$R(\pi) \xrightarrow{I} SL(2,\mathbb{C})^n$$

$$\rho \longmapsto (\rho(\gamma_1), \rho(\gamma_2), \dots, \rho(\gamma_n))$$

is a topological embedding. [i.e I is a open continuous map]

[follows directly from the def of cpt-open top]

Eg. $R(\mathbb{Z}) = R(\langle t \rangle) \cong SL(2,\mathbb{C})$

$\rho(t) =$ any element in $SL(2,\mathbb{C})$

$R(F_n) \cong SL(2,\mathbb{C})^n$

$\rightsquigarrow$ $R(\Pi)$ is metrisable & $\lim \rho_k = \rho$ iff $\lim \rho_k(\gamma_i) = \rho(\gamma_i)$ $\forall i = 1, \cdots, n$

Relator r_i = a word in $\gamma_1, \cdots, \gamma_n$

$\gamma_1 \sim A_1 \in SL(2,\mathbb{C})$, $\gamma_2 \sim A_2 \in SL(2,\mathbb{C})$, $\cdots$, $\gamma_n \sim A_n \in SL(2,\mathbb{C})$

then $r_i \sim$ some element in $SL(2,\mathbb{C})$

$r_i : SL(2,\mathbb{C})^n \to SL(2,\mathbb{C})$

$(A_1, \cdots, A_n) \mapsto r_i(A_1, \cdots, A_n)$ $\leftarrow$ change appearance of γ_i by A_i

is a cts (polynomial) map

i.e. $r_2 = \gamma_1^{-1}\gamma_n^3\gamma_2^2$, then $r_2(A_1, \cdots, A_n) = A_1^{-1}A_n^3A_2^2$

So by the first isomorphism of group:

$(A_1, \cdots, A_n) \in \tau(R(\Pi))$ iff $r_i(A_1, \cdots, A_n) = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\forall i \in I$

Hence, we have a bijection.

$$R(\pi) \equiv Z(R(\pi)) = \{ (A_1, \cdots, A_n) \in SL(2,\mathbb{C})^n : \; r_i(A_1, \cdots, A_n) = Id, \; \forall i \in I \}$$

$$\rho \sim (\rho(\gamma_1), \cdots, \rho(\gamma_n))$$

Write $A_j = \rho(\gamma_j) = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, then $A_j^{-1} = \begin{pmatrix} d_j & -b_j \\ -c_j & a_j \end{pmatrix}$

$$\Rightarrow \; r_i(A_1, \cdots, A_n) = \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix}$$

$$p_i, q_i, r_i, s_i \in \mathbb{Z}[a_j, b_j, c_j, d_j]_{j=1}^{n}$$

$$R(\pi) \equiv \{ (a_j, b_j, c_j, d_j)_{1 \le j \le n} \in \mathbb{C}^{4n} : \; a_j d_j - b_j c_j = 1, \; \forall j \;\&\; \begin{matrix} p_i = s_i = 1 \\ q_i = r_i = 0 \end{matrix} \; \forall i \in I \}$$

$$\subseteq \mathbb{C}^{4n}$$

$R(\pi) \equiv Z(R(\pi)) \subset SL(2,\mathbb{C})^n \subset \mathbb{C}^{4n}$

is the zero set of a <u>family of polynomials</u> in $4n$ (complex) variables (with $\mathbb{Z}$-coefficients)

$\longleftarrow$ Hilbert Basis Theorem

a family of finitely many polynomials

Prop: $R(\pi)$ is naturally identified with a complex affine algebraic set $\subset \mathbb{C}^{4n}$.

Moreover, each $\gamma \in \pi$, $e_\gamma : R(\pi) \to SL(2,\mathbb{C}) \subset \mathbb{C}^4$
$\rho \mapsto \rho(\gamma)$

is a <u>regular function</u>.
↳ a polynomial function

i.e.

[$e_\gamma(\_) = (e^1_\gamma(\_), e^2_\gamma(\_), e^3_\gamma(\_), e^4_\gamma(\_))$

each $e^i_\gamma \in \mathbb{C}[R(\pi)]$ ← coordinate ring]

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$R(\pi)$: cpt-open top or <u>a complex affine variety with Zariski top</u> ← algebro-geometric invariants.

Rmk: ①: this setup readily generlizes to any complex affine algebraic Lie groups. G.

$SL(n,\mathbb{C})$, $SU(2)$, $PSL(2,\mathbb{C})$.

[$Ad: PSL(2,\mathbb{C}) \to Aut(sl_2(\mathbb{C})) < SL(3,\mathbb{C})$]

↑ adjoint representation ↑ Lie algebra of dim 3

②: different presentations of π lead to isomorphic algebraic sets $R(\pi)$

Def $\rho \in R(\pi)$ i.e $\rho : \pi \to SL(2, \mathbb{C})$

is <u>reducible</u> if its image is conjugated into a group of upper triangular matrices.

($\begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftarrow \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is a fixed subspace)

<u>irreducible</u> otherwise.

Prop [Culler, Shalen] $\rho \in R(\pi)$

ρ reducible $\iff \operatorname{tr}(\rho(\gamma)) = 2 \quad \forall \gamma \in [\pi, \pi]$

Def: the <u>character</u> of $\rho \in R(\pi)$ is

$$\chi_\rho : \pi \to \mathbb{C}$$
$$\gamma \mapsto \operatorname{tr}(\rho(\gamma))$$

So if $\rho_1 = A \rho_2 A^{-1}$, ~~or ρ_1~~ then $\chi_{\rho_1} = \chi_{\rho_2}$

Prop [Culler, Shalen] ρ_1 irreducible.

$\chi_{\rho_1} = \chi_{\rho_2} \implies \rho_1, \rho_2$ are conjugate

[i.e within irreducible repn, $\rho_1 \sim \rho_2 \iff \chi_{\rho_1} = \chi_{\rho_2}$]

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Rmk: not true for reducible repns.

$\mathbb{Z} = \langle t \rangle \to SL(2,\mathbb{C})$, $\rho_1(t) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\rho_2 = \begin{pmatrix}1&1\\0&1\end{pmatrix}$

$\chi_{\rho_1} = \chi_{\rho_2}$ but $\rho_1 \not\sim \rho_2$

$X(\pi)$ = the set of characters of all $\rho \in R(\pi)$

$\uparrow$
no top yet.

— $\forall \gamma \in \pi$, evaluation map

$$I_\gamma : X(\pi) \to \mathbb{C}, \quad \chi_\rho \mapsto \chi_\rho(\gamma)$$

So if $\gamma_2 \sim \gamma_1^{\pm 1}$, then $I_{\gamma_1} = I_{\gamma_2}$

& $I_{\gamma_1\gamma_2} + I_{\gamma_1\gamma_2^{-1}} = I_{\gamma_1} I_{\gamma_2}$ [in $SL(2,\mathbb{C})$, $tr(x)tr(y) = tr(xy) + tr(xy^{-1})$]

Fact: ① $I_{\gamma^n} = f(I_\gamma)$ $\quad f(x) \in \mathbb{Z}[x]$

② $I_{[\gamma_1,\gamma_2]} = I_{\gamma_1}^2 + I_{\gamma_2}^2 + I_{\gamma_1\gamma_2}^2 - I_{\gamma_1} I_{\gamma_2} I_{\gamma_1\gamma_2} - 2$

Prop: $\pi = \langle \gamma_1, \gamma_2 \mid r_i,\ i \in I \rangle$

Then $\forall \gamma \in \pi$ $\exists P_\gamma \in \mathbb{Z}[x,y,z]$ s.t.

$I_\gamma = P_\gamma(I_{\gamma_1}, I_{\gamma_2}, I_{\gamma_1\gamma_2})$. Hence we have an embedding $X(\pi) \to \mathbb{C}^3$

Hence, we have an embedding

$$X(\pi) \to \mathbb{C}^3,\ \chi_\rho \mapsto (I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$$

(Pf.) $\gamma \sim \gamma_2^n$ ✓

$\gamma \sim \gamma_1^{a_1}\gamma_2^{b_1}\cdots\gamma_1^{a_n}\gamma_2^{b_n}$. induction on $\Sigma\, |a_i| + |b_i|$

This result can be generalized to

$$\pi = \langle \gamma_1, \cdots, \gamma_n \mid r_i,\ i \in I \rangle$$

$\forall \chi \in X(\pi)$ is determined by its value on the finite set $\{\gamma_{i_1}\cdots\gamma_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$

$\rightsquigarrow$ embedding $X(\pi) \to \mathbb{C}^p$

$$\chi \mapsto (I_{\gamma_1\cdots\gamma_k}(\chi_\rho))$$

Prop: [Vogt 89]. χ is determined by

$$\{\gamma_i\} \cup \{\gamma_{i_1}\gamma_{i_2} : 1 \le i_1 < i_2 \le n\} \cup \{\gamma_{i_1}\gamma_{i_2}\gamma_{i_3} : 1 \le i_1 < i_2 < i_3 \le n\}$$

$$p = \min\left\{\frac{n^2(n+5)}{6},\ 2^n - 1\right\}$$

Eg. $\pi = F_2$, then $X(\pi) \to \mathbb{C}^3$ is a bijection

Try to equip $X(\pi)$ with the structure of a complex affine variety.

$t: R(\pi) \to X(\pi)$, $\rho \mapsto \chi_\rho$

for $\forall \gamma \in \pi$, $tr(e_\gamma(\rho)) = \chi_\rho(\gamma) = I_\gamma(t(\rho))$

$e_\gamma: R(\pi) \to SL(2,\mathbb{C}) \subset \mathbb{C}^4$ is a regular function

$\Rightarrow$ $tr \circ e_\gamma$ is also a regular function

$$R(\pi) \xrightarrow{t} X(\pi) \subset \mathbb{C}^p$$

$$\rho \mapsto \chi_\rho \equiv (I_{\gamma_{i_1} \cdots \gamma_{i_k}}(\chi_\rho))$$

t: is a regular function

Thm. $X(\pi) = t(R(\pi)) \subseteq \mathbb{C}^p$ is an algebraic set

[Culler, Shalen]

Rmk: $X(\pi)$ depends on the choice of presentation of π, but well-defined up to a canonical isomorphism of algebraic sets $\boxtimes$

Def. the <u>$SL(2,\mathbb{C})$-character variety</u> of π is $X(\pi)$, endowed with this structure

[Eg. $X(F_2) = \mathbb{C}^3 \leftarrow$ affine space]

Prop. $K \subset S^3$ knot. $M_K = \overline{S^3 - n(K)}$
$\pi_1(M_K) = G$.

write $X(M_K) = X(G)$

then $X^{red}(M_K) \cong \mathbb{C}$

Pf: $\rho \in R(\pi)$ reducible

$\rho \sim \rho'$: $\rho(\gamma) = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$ $\forall \gamma \in \pi$

define $\rho_{ab}(\gamma) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ $\forall \gamma \in \pi$

then ρ_{ab} is an abelian repn. & $\chi_\rho = \chi_{\rho'} = \chi_{\rho_{ab}}$

abelian reprs factor through

$G \to G/[G,G] \cong \langle t \rangle \to SL(2,\mathbb{C})$

$\rho_{ab}(t) = \text{any} \in SL(2,\mathbb{C})$

$\chi_{\rho_{ab}} \sim \text{any} \in \mathbb{C}$.

Def $\quad X^{irr}(\pi_1) = \overline{X(\pi_1) \setminus X^{red}(\pi_1)} \subseteq X(\pi_1)$

[$X^{irr}(\pi_1)$ may be [in most cases] is reducible,
i.e a union of two proper algebraic subsets]

Eg. $\quad K =$ trefoil knot

$\pi_1(M_K) = \langle \gamma_1, \gamma_2 \mid \gamma_1^2 = \gamma_2^3 \rangle$

$\gamma_1^2 = \gamma_2^3$ is central in $\pi_1(M_K)$

$\rho \in R(M_K)$ irreducible $\Rightarrow \rho(\gamma_1^2 = \gamma_2^3) = \pm I$

$\Rightarrow (\rho(\gamma_1))^2 = (\rho(\gamma_2))^3 = \pm I$

$\rho(\gamma_1) \neq -I$, or ρ reducible

$\rightarrow \rho(\gamma_1)$ has order 4, i.e. $\rho(\gamma_1) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\rho(\gamma_2)^3 = \rho(\gamma_1)^2 = -I$.

$\rightarrow \rho(\gamma_2)$ has order 6, $\rho(\gamma_1) \sim \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

$\left(\sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \theta = \frac{\pi}{3} \right)$

$X^{irr}(M_K)$ under the embedding $X(\pi) \to \mathbb{C}^3$

looks like $\quad \chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \chi(\gamma_1\gamma_2))$

$$\mathrm{image}(X^{irr}(M_K)) \subset \{(0,1,w) : w \in \mathbb{C}\}$$

Indeed, define $\rho_z : \pi_1(M) \to SL(2,\mathbb{C})$

$$\gamma_1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\gamma_2 \mapsto \begin{pmatrix} z & -(z^2 - z + 1) \\ 1 & 1 - z \end{pmatrix}$$

then $\chi_{\rho_z}(\gamma_1\gamma_2) = 2iz - i$

so $\mathrm{image}(X^{irr}(M_K)) = \{(0,1,\omega) : w \in \mathbb{C}\}$

$$X(M_K) = X_0 \cup X_1 \to X^{red} \cong \mathbb{C}$$

$$X_0 \swarrow X^{irr} = \{(0,1,\omega) : \omega \in \mathbb{C}\}$$

Slogan: make a trade-off at the expense of some loss of information and construct a useful & computable invariant.

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$K \subset S^3$ a non-trivial knot

$M_K = \overline{S^3 - n(K)}$

Then $\partial n(K) = \partial M_K =$ torus

$i: \partial M_K \to M_K \rightsquigarrow i_*: \pi_1(\partial M_K) \to M_K$ embedding

$\pi_1(\partial M_K) = \mathbb{Z} \oplus \mathbb{Z} = \langle \mu, \lambda \rangle$ meridian & longitude.

$\rightsquigarrow i_\#: X^{irr}(M_K) \to X(\partial M)$ is ~~an algebraic set~~ ~~morphism~~ regular functi.

$$\chi_\rho \mapsto \chi_{\rho \circ i_*}$$

$\to \overline{i_\#(X^{irr}(M_K))}$ is an algebraic set

$\rho \in R(\partial M)$ has abelian image, hence reducible

$\rho \sim \rho'$ $\quad \rho'$ has image in $\left\{ \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}$

$\chi_\rho = \chi_{\rho'} = \chi_{\rho^{ab}}$ $\quad \rho^{ab}(\gamma) = \mathrm{diag}(\rho'(\gamma))$

$\Rightarrow$ each $\chi \in X(\partial M)$ is the character of some representation with diagonal images.

Def. $D(\partial M) \subset R(\partial M)$

$= \{\rho \in R(\partial M) : \text{image}(\rho) \text{ is diagonal}\}$

Then $t_0 : D(\partial M) \to X(\partial M)$ is surjective

$\rho \mapsto \chi_\rho$

t_0 is the restriction of $t : R(\partial M) \to X(\partial M)$

so is a morphism of algebraic sets

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$\eta : D(\partial M) \to \mathbb{C}^2$

$\rho \mapsto (m, l)$. $\quad \rho(\mu) = \begin{pmatrix} m & \\ & m^{-1} \end{pmatrix}$

$\rho(\lambda) = \begin{pmatrix} l & \\ & l^{-1} \end{pmatrix}$

$X_0 \subset X^{irr}(M)$: 1-dim irreducible algebraic set

intuitive "dim"

$$\begin{array}{ccccc} D(\partial M) \supset W_0 = t_0^{-1}(\xi_0) & \xrightarrow{\eta|_{W_0}} & \eta(W_0) \subset \overline{\eta(W_0)} \subset \mathbb{C}^2 \\ \downarrow t_0 & & \| \\ & & D_0 \end{array}$$

$$X_0 \longrightarrow Y_0 = \overline{i_\#(X_0)} \subset X(\partial M)$$
$$\cap$$
$$X^{irr}(M_K)$$

D_0 is a plane curve = the zero set of a 2-variable complex polynomial $A_{X_0}(m, l)$

Def A-polynomial $A_K(m,l) = \prod_{X_i} A_{X_i}(m,l)$

X_i 1-dim irreducible $\subset X^{irr}$

Fact: Up to a multiplication, $A_K(m,l) \in \mathbb{Z}[m,l]$

Thm. (Boyer, Zhang 2005, Dunfield, Garoufalidis 2005)
K nontrivial knot $\Rightarrow A_K(m,l)$ is non-trivial