Orderable Groups and Bundles.

G-sets (G a group).

Def: A G-set is a set X together with a right action of G, ie.
$$X \times G \longrightarrow X$$
$$(x, g) \longmapsto x \cdot g$$

Examples: Starting with a group G, we get:

- G_r, the G-set whose underlying set is G, with right mult. as the action:
$$(h, g) \longmapsto hg \leftarrow \text{product in } G.$$

- If G is a group having a natural left action on a set (e.g. a group of bijections $f: X \to X$) then we get a G-set by
$$x \cdot f = f^{-1}(x).$$
Then e.g. $x \cdot (fg) = g^{-1}f^{-1}(x) = (x \cdot f) \cdot g$, so we can change from left to right if need be.

- Can make a $(G \times G)$-set ${}_\ell G_r$ with underlying set G and action
$$f \cdot (g, h) = g^{-1} f h \quad (\text{product in } G).$$

- If G is a group of functions $g: X \to X$, then we can make a G-set via
$$x \cdot g = g^{-1}(x)$$
Then note $(x \cdot g) \cdot h = h^{-1}(g^{-1}(x)) = h^{-1}g^{-1}(x) = x \cdot gh$. ✓

We can also define maps between G-sets, by taking G-equivariant set maps

$$f: X \longrightarrow Y$$

with $f(x \cdot g) = f(x) \cdot g \ \forall x \in X, g \in G.$

Ex: Suppose N is a normal subgroup of G. Then ${}_N G = \{Ng \mid g \in G\}$

is a G-set since $Ng \cdot h = Ngh$ defines an action. Then

$$h \longmapsto Nh$$

is a map between the G-sets G_r and ${}_N G$.

Rmk: With G-sets as objects and G-set maps as morphisms, we get a category

<u>G-Set</u>.

Covering spaces:

Def: A cts map $p: E \to B$ is a covering map if $\forall x \in B \ \exists$ nbhd U_x s.t. $p^{-1}(U_x) = \bigcup_{i \in I} V_i$, where V_i are disjoint open subsets of E s.t. $p|_{V_i}: V_i \to U_x$ is a homeomorphism.

We're going to use "pointed spaces" $(E, e_0), (B, b_0)$ with $p(e_0) = b_0$, for reasons that will soon clear up.

- Ex:

$$p: \mathbb{R} \longrightarrow S^1, \quad p(t) = (\cos(2\pi t), \sin(2\pi t)).$$

- If (B, b_0) is "nice enough", e.g. connected and locally simply connected, then $\exists$ a universal covering space. Here is a reminder of how it's built:

Fix $b_0 \in B$, set

$$\tilde{B} = \{\alpha : [0,1] \longrightarrow B \mid \alpha(0) = b_0\}/\sim$$

where $\sim$ is homotopy of paths fixing endpoints.

Topologizing $\tilde{B}$ is a bit of a mess, so let me skip that. But we get

$$p : \tilde{B} \longrightarrow B \qquad p([\alpha]) = \alpha(1) \text{ a covering map.}$$

The path $\alpha(t) = b_0$ gives $[\alpha] \in \tilde{B}$ with $p([\alpha]) = b_0$, set $e_0 = [\alpha]$.

- We can also make more general "path space"

$$\mathrm{Path}(B) = \{\alpha : [0,1] \longrightarrow B\}/\sim$$

where $\sim$ is homotopy fixing endpoints. Topologize $\mathrm{Path}(B)$ using the same technique, define

$$p : \mathrm{Path}(B) \longrightarrow B \times B \text{ by}$$
$$p([\alpha]) = (\alpha(0), \alpha(1))$$

and get a covering map as before.

We again take our basepoint in $\mathrm{Path}(B)$ as the (equiv class of) the constant map $\alpha(t) = b_0 \ \forall t \in [0,1]$, and base point of $B \times B$ is (b_0, b_0).

For a fixed space B, we can also define maps of covering spaces to be cts $f : E_1 \longrightarrow E_2$ such that

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr] & & E_2 \arrow[dl] \\ & B & \end{tikzcd}$$

commutes.

Rmk: Fixing a space B, with covering spaces as objects and covering maps as arrows, we get a category $\underline{\mathrm{Cov}(B)}$.

Fundamental idea of covering spaces:

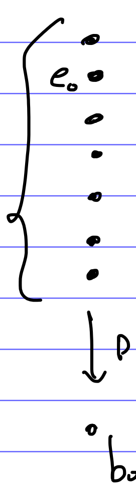
Thm: If B is a space having a universal cover, and $G = \pi_1(B, b_0)$, then there is an equivalence of categories

$$\begin{tikzcd} \underline{G\text{-Set}} \arrow[r, bend right, "u"'] & \mathrm{Cov}(B) \arrow[l, bend right, "F"'] \end{tikzcd}.$$

Pf: Let us "see" the correspondence, at least at the level of objects.

Given

$$\begin{tikzcd} (E, e_0) \arrow[d, "p"'] \arrow[r, "F"] & F(E) = p^{-1}(b_0) \\ (B, b_0) & \end{tikzcd}$$



Then $p^{-1}(b_0)$ becomes a G-set as follows:
Given $e \in p^{-1}(b_0)$, $[\gamma] \in G = \pi_1(B, b_0)$,

Lift γ to get $\tilde{\gamma}:[0,1] \longrightarrow E$ with $\tilde{\gamma}(0)=e$. Define

$$e \cdot [\gamma] = \tilde{\gamma}(1),$$

which is again an element of $p^{-1}(b_0)$ since $\gamma(0) = \gamma(1) = b_0$. Fund. result of covering spaces is that this is indep. of choice of γ, ie $\underset{\substack{\uparrow \\ \text{homotopic}}}{\gamma \sim \gamma'} \Longrightarrow \tilde{\gamma}(1) = \tilde{\gamma}'(1)$.

Conversely, if suppose $\tilde{B}$ exists and X is a G-set. Then there is an action of G on $\tilde{B}$ by deck transformations, ie $\forall g \in G$ $\exists$ $\varphi_g : \tilde{B} \longrightarrow \tilde{B}$ s.t. $p(\varphi_g(e)) = p(e)$ $\forall e \in \tilde{B}$.

We can build a covering space $U(X)$ by

$$U(X) = (X \times \tilde{B})/\sim, \text{ where } (x,e) \sim (x \cdot g, \varphi_g(e))$$

and then make a map $q : U(X) \longrightarrow B$ by

$q(x,e) = p(e)$, where $p: \tilde{B} \longrightarrow B$ is the univ. cover.

The top on $X \times \tilde{B}$ is product, on $(X \times \tilde{B})/\sim$ is quotient.

We can also make F and U act on maps:

If
$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$
is a map of covering spaces, then f maps $p_1^{-1}(b_0)$ to $p_2^{-1}(b_0)$, and one can check that $\forall e \in p_1^{-1}(b_0)$

$$f(e) \cdot g = f(e \cdot g) \quad \forall g \in \pi_1(B) \quad \text{(compose your lift with } f\text{)}$$

$\Rightarrow$ $F(f) : F(E_1) \longrightarrow F(E_2)$ is $F(f)(e) = f(e)$

$\Rightarrow F(f)(e \cdot g) = f(e \cdot g) = f(e) \cdot g$
$= F(f)(e) \cdot g$.

And if $g : X_1 \longrightarrow X_2$ a map of G-sets, then $U(g) : U(X_1) \longrightarrow U(X_2)$ is

$$U(g)([x,e]) = [g(x), e]$$

Then check this is a covering space map //

<u>Moral</u>: Any time you have a bunch of G-sets with maps

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd} , \quad \xrightarrow{U} \quad \begin{tikzcd} U(X) \arrow[r] \arrow[d] & U(X') \arrow[d] \\ U(Y) \arrow[r] & U(Y') \end{tikzcd}$$

you get some corresponding collection of maps between covering spaces.

Orderable groups:

Def: A group G is right orderable if $\exists$ a strict total order $<$ of G s.t.

$g<h \Rightarrow gf<hf \quad \forall f,g,h \in G$.

Rmk: These are all ~~torsion~~-free, since

$1<g \Rightarrow g<g^2 \Rightarrow 1<g<g^2<\ldots$ etc.

In particular, they're infinite.

E.g: $\mathbb{Z}$, $\mathbb{Z}^2$, free groups, braid groups, some matrix groups, torsion-free nilpotent groups.

Or, alternatively:

Thm: G countable. Then G is right-orderable iff $\exists$ an embedding

$$G \xhookrightarrow{\varphi} \mathrm{Homeo}_+(\mathbb{R}).$$

Or, as G-sets; we can make $\mathbb{R}$ into a G-set:

$$x \cdot g = \varphi(g^{-1})(x)$$

Then we get a theorem

Thm: G is right-orderable iff $\exists$ a right G-action on $\mathbb{R}$ and an inclusion

$$i : G_r \hookrightarrow \mathbb{R}$$

of G-sets that respects the orders (on G and $\mathbb{R}$)

So, what we would like to do is "import to covering spaces" using the functor U, get some theorem like:

<u>Thm</u>: $\pi_1(B, b_0)$ is right-orderable iff .... ?

<u>Hiccup</u>: (Sweeping a lot under the rug)

$\mathbb{R}$ is <u>not</u> a discrete space, which should be the case if we want a covering space with this space as fibre. So while I used $U: G\text{-Set} \longrightarrow \mathrm{Cov}(B)$ as motivation, here you need to use

$$U: G\text{-Space} \longrightarrow LCB(B)$$

(locally constant bundles over B)

With this fix,

$$\begin{tikzcd} & \underline{G\text{-Space}} & & \underline{LCB}(B) & \\ G \text{ RO} \arrow[r, phantom, "\iff"] & G_r \arrow[d] \arrow[rr, "U"] & & \tilde{B} \arrow[d] \arrow[r, phantom, "\iff"] & \pi_1(B,b_0) \text{ is RO} \\ & \mathbb{R} & & U(\mathbb{R}) & \end{tikzcd}$$

Ie, $\pi_1(B, b_0)$ is RO iff $\exists$ some bundle $U(\mathbb{R})$ s.t. $\tilde{B} \hookrightarrow U(\mathbb{R})$.

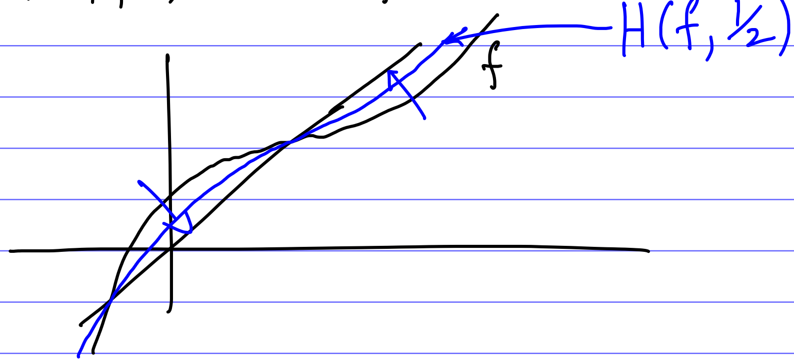
In fact,

This can be skipped

Lemma: Equipped with compact open topology, $\mathrm{Homeo}_+(\mathbb{R})$ is contractible.

Pf: Deform $f(x)$ to id by "straight lines"

$$H(f,t) = (1-t)f(x) + tx$$



Corollary: $U(\mathbb{R})$ is the trivial bundle, ie $U(\mathbb{R}) \simeq B \times \mathbb{R}$.

Theorem: (Farrell). $G = \pi_1(B, b_0)$

The group G is RO iff $\exists$ an embedding $h : \tilde{B} \hookrightarrow B \times \mathbb{R}$ s.t.

$$\begin{tikzcd} \tilde{B} \arrow[rr, "h"] \arrow[dr, "p"'] & & B \times \mathbb{R} \arrow[dl, "\pi_1"] & U(\mathbb{R}) \\ & B & & \end{tikzcd}$$

Rmk: This perspective proves one direction, namely G LO $\Longrightarrow \exists\, h$, the other direction is a bit of fuss.

Here's why I like this perspective: We can prove that G countable and BO iff

$${}_{\ell}G_r \hookrightarrow \mathbb{R}$$, again, order-preserving.
(category of $G \times G$-spaces?)

Apply all the same machinery. Then
(Anel, C)

<u>Thm</u>: $G = \pi_1(B, b)$ iff $\exists$ an embedding
$$h : P(B) \longrightarrow (B \times B) \times \mathbb{R}$$

s.t.

$$\begin{tikzcd}
P(B) \arrow[rr] \arrow[dr, "p"'] & & (B \times B) \times \mathbb{R} \arrow[dl, "\pi_1"] \\
& B \times B &
\end{tikzcd}$$

path space example from earlier.

<u>Q</u>: I bet there are other theorems to be found this way. E.g.

G Archimedean ordered $\Longleftrightarrow \exists\, G \hookrightarrow (\mathbb{R}, +)$

Can we write this as a fact about G-spaces which has a topological counterpart about bundles?