Geometric Mechanics, Integrability, and the Kepler Problem

Geometry & Topology Learning Seminar

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October 7, 2024

Contents

Introduction to Geometric Mechanics

Historical Introduction

Mechanics is the subfield of physics which studies motion. There are multiple different approaches to mechanics: Newtonian, Hamiltonian, Lagrangian, Hamilton-Jacobi,...

- (1686) The first rigorous formulation of mechanics was developed by Isaac Newton (Newton's Laws) and contemporaries (e.g. Hooke).
- (1686-1690) Newton's formulation of mechanics allows for a first-principles derivation of Kepler's laws of orbital motion.
- (1760) Lagrange presents the 'principle of least action', inventing the Lagrangian formulation of mechanics, in terms of second order ODEs.
- (1833) Hamilton develops the Hamiltonian formulation of mechanics, in terms of first order ODEs.
- (1884) Jacobi extends the work of Hamilton, reformulates mechanics in terms of a PDE.
- (1900s-Present) Lagrangian, Hamiltonian, Hamilton-Jacobi formulations are all described with differential geometry.

Kepler's Laws

1. The orbit of a planet around the sun is an ellipse, with the sun at one focus.

$$r(\theta) = \frac{p}{1 + \epsilon \cos(\theta)} \tag{1}$$

2. Given any time interval Δt, the line segment joining the planet to the sun sweeps out an area ΔA, independently of the starting position of the planet.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mathrm{Const} \tag{2}$$

3. The square of the period of the orbit is proportional to the cube of the semi-major axis of the ellipse.

$$P^2 \propto a^3 \tag{3}$$

Key point: qualitative description of orbits in terms of few parameters.

Generalized Kepler Problem

- How is the Kepler problem formulated mathematically as an initial value problem (IVP)?
- How can the methods used for this be generalized to other interesting systems?
- Can we classify which dynamical systems have bounded, periodic orbits?
- Can we qualitatively describe these systems without solving the IVP?

Basic Definitions

Definition (Configuration Space)

The space of all physical configurations of a system is called the **configuration space**. Typically, it is a smooth manifold we call M.

Example

Consider a point particle in three-dimensional Euclidean space. The configuration space is just the set of all possible positions $M = \mathbb{R}^3$.

Remark: In this presentation we will only consider a single particle moving in a configuration space M.

Remark: This notion can be generalized to infinite dimensional manifolds (Banach manifolds, Frechet manifolds,...) in order to describe fields such as the electromagnetic field.

Tangent and Cotangent Bundles

Definition (Tangent and Cotangent Bundle)

The **tangent bundle** TM of a smooth manifold M is the disjoint union of all tangent hyperplanes to M.
The **cotangent bundle** T^*M is the disjoint union of their dual spaces.

Definition (Vector Field)

A vector field is a smooth function $v : M \to TM$ such that $v(p)$ lies in the tangent space at p, T_pM. We say $v \in \mathfrak{X}(M)$.

Definition (Differential Form)

A differential 1-form is a smooth function $\alpha : M \to T^*M$ such that $\alpha(p)$ lies in the dual space to the tangent space at p, T_p^*M. We say $\alpha \in \Omega^1(M)$.

Remark: A differential 1-form can be applied to a vector field to get a scalar function, $\alpha(v) \in C^\infty(M)$.

Vector Fields

Definition (Natural Action of Vector Field)

There is a natural action of vector fields $\mathfrak{X}(M)$ on $C^\infty(M)$, where $v \cdot f$ is the directional derivative of f in the direction of v.

Remark: This action uniquely determines $\mathfrak{X}(M)$ up to isomorphism (i.e. any vector field is defined by its action on an arbitrary function).

Theorem (Basis for T_pM)

Given a coordinate patch $(x^1, ..., x^n)$, $U \subseteq M$, a basis for T_pM is given by $\{\partial_{x^1}|_p, ..., \partial_{x^n}|_p\}$, which are the partial derivative operators in each direction.

Theorem (Vector fields form a Lie algebra)

The space $\mathfrak{X}(M)$ of vector fields forms a Lie algebra, where the Lie bracket is defined by

$$[v, w] \cdot f = v \cdot (w \cdot f) - w \cdot (v \cdot f)$$

Products of Forms

Definition (Tensor Product)

Given differential forms, $\alpha_1, ..., \alpha_k$, we can take their tensor product,

$$\alpha_1 \otimes ... \otimes \alpha_k(v_1, ..., v_k) = \alpha_1(v_1)\alpha_2(v_2)...\alpha_n(v_k)$$

Definition (Wedge Product)

Given differential forms $\alpha_1, ..., \alpha_k$ we can take their wedge product,

$$\alpha_1 \wedge ... \wedge \alpha_k(v_1, ..., v_k) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\alpha_1(v_{\sigma(1)})...\alpha_n(v_{\sigma(k)})$$

We call such an object a differential k-form

Remark: The space of all differential k-forms is called $\Omega^k(M)$.
Remark: $\Omega^k(M) = 0$ for $k > \dim(M)$ and $\Omega^0(M) = C^\infty(M)$.
Remark: The space $\Omega^\bullet(M) = \bigoplus \Omega^k(M)$ forms a graded algebra.

Exterior Derivative

Definition (Exterior Derivative)

The exterior derivative is the unique linear map $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$, sending $\Omega^k(M) \mapsto \Omega^{k+1}(M)$, satisfying the rules:

- Given $f \in \Omega^0(M) = C^\infty(M)$, we define $\mathrm{d}f(v) = v \cdot f$ for all $v \in \mathfrak{X}(M)$.
- For all $\alpha \in \Omega^k(M)$, $\beta \in \Omega^\ell(M)$, we have

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$$

Remark: This makes $\Omega^\bullet(M)$ into a differential graded (dg) algebra.

Theorem (Basis for T_p^*M)

Given a coordinate patch $(x^1, ..., x^n)$, $U \subseteq M$, a basis for T_p^*M is given by $\{\mathrm{d}x^1|_p, ..., \mathrm{d}x^n|_p\}$.

Remark: This is the same as the Kronecker dual basis for $\{\partial_{x^1}|_p, ..., \partial_{x^n}|_p\}$.

General Tensors

Definition (Tensor)

A **tensor field** of type (k, ℓ) is a smooth functional $T : T^*M^k \times TM^\ell \to \mathbb{R}$.

Remark: Most of the time in physics, this is what people mean by tensor.
Remark: Differential k-forms are totally antisymmetric tensors of type $(0, k)$.

Definition (Contraction)

Let v be a vector field and T be a (k, ℓ) tensor. We define the **interior product** or **contraction**, $v \lrcorner T$ to be the $(k, \ell - 1)$ tensor defined by,

$$v \lrcorner T(\alpha_1, ..., \alpha_k, v_1, ..., v_{\ell-1}) := T(\alpha_1, ..., \alpha_k, v, v_1, ..., v_{\ell-1})$$

$\alpha \lrcorner T$ is defined similarly for 1-forms.

Remark: Some people use the notation $\iota_v T$ instead.

(pseudo-)Riemannian Geometry

Definition (Metric)

A **metric** is a non-degenerate symmetric bilinear functional on TM. That is, for any vector fields v, w,

- $g(v, w) = g(w, v)$ (Symmetric)
- If $g(v, w) = 0$ for all w, then $v = 0$ (Non-Degenerate)

Remark: A metric is symmetric, so it is not a differential 2-form.

Definition (Musical Isomorphisms)

Given a metric g, we have an isomorphism $TM \cong T^*M$ given by,

$$\flat : v \mapsto v^\flat := g(v, -) = \iota_v g = v \lrcorner g \tag{4}$$

Given a local frame e_i for TM and dual frame e^i ($e^i(e_j) := \delta^i{}_j$) for T^*M we have

$$v^\flat = g_{ij} v^j e^i \tag{5}$$

Velocity and Acceleration

Definition

Let M be a smooth manifold and $q : I \to M$ a smooth curve. The **velocity** of the curve is defined as $v = \dot{q} := \frac{\partial q}{\partial t} \in T_{q(t)}M$.

Definition

The **acceleration** of the curve is defined as $a = \ddot{q} := \nabla_{\dot{q}}\dot{q} \in T_{q(t)}M$.
For $M = \mathbb{R}^n$ with the standard metric $g = \sum \mathrm{d}x^i \otimes \mathrm{d}x^i$, this reduces to $\ddot{q} = \frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$.

Remark: A particle travelling along a geodesic experiences no acceleration (Einstein's Principle of Equivalence).

Momentum

Definition (Momentum)

Let $q : I \to M$ be a curve representing the position of a particle in a manifold M. The **momentum** is defined as $p \in T^*_{q(t)}M$, where

$$p := m\dot{q}^\flat \tag{6}$$

Where m is the mass of the particle.

Remark: This is only valid for $m \neq 0$.

Hamiltonians

Definition (Hamiltonian)

A **Hamiltonian** is a smooth function $H : T^*M \to \mathbb{R}$, which represents the 'total energy' of the system.

Definition (Hamilton's Equations)

Given a Hamiltonian H, we may define a dynamical system, represented by $(q(t), p(t))$, on the manifold T^*M according to Hamilton's Equations:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p^i} = \mathrm{d}H(\partial_{p^i}) \tag{7}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} = -\mathrm{d}H(\partial_{q^i}) \tag{8}$$

Remark: The flow Φ_H defined by this dynamical system can be written as the flow of a vector field X_H, called the Hamiltonian vector field.

Example: Kepler Hamiltonian

Example (Kepler system in $\mathbb{R}^3$/Reduced two-body problem)

Consider a particle with position $\mathrm{q} \in \mathbb{R}^3$ and momentum $\mathrm{p} \in T^*_{\mathrm{q}}\mathbb{R}^3 \cong \mathbb{R}^3$. The kinetic energy K is defined as,

$$K = \frac{1}{2m}\|\mathrm{p}\|^2$$

In a gravitational field, a particle also has **potential energy**, modelled for example by the function

$$U = -\frac{k}{\|\mathrm{q}\|}$$

The Hamiltonian function is then

$$H = K + U = \frac{1}{2m}\|\mathrm{p}\|^2 - \frac{k}{\|\mathrm{q}\|}$$

Any Hamiltonian of the form $K(p) + U(q)$ is called separable.

Phase Space is Symplectic

Definition (Phase Space)

Let M be the configuration space of a system. The **phase space** is the cotangent bundle T^*M.

Definition (Canonical Symplectic Form)

Let M be an n dimensional configuration space. The **canonical symplectic form** $\Omega \in \Omega^2(T^*M)$ is the closed non-degenerate 2-form on T^*M defined locally by,

$$\Omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p^i \tag{9}$$

Here, p^i are the coordinates on the fiber T_x^*M induced by applying the musical isomorphism to $m\partial_{q^i}$ for each $i = 1, ..., n$.

Remark: Since Ω is non-degenerate, the map $X \mapsto \Omega(X, -)$ defines an isomorphism $\tilde{\Omega} : \mathfrak{X}(T^*M) \to \Omega^1(T^*M)$.

Poisson Bracket

Definition (Hamiltonian Vector Field)

Given any function $f : T^*M \to \mathbb{R}$, we define a vector field $X_f = \tilde{\Omega}^{-1}(\mathrm{d}f)$ called the Hamiltonian vector field generated by f.

Definition (Poisson Bracket)

Let $f, g : T^*M \to \mathbb{R}$ be differentiable. Then the Poisson bracket of f and g is defined by,

$$\{f, g\} = \Omega(X_f, X_g) = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} \tag{10}$$

The Poisson bracket is antisymmetric, bilinear, and satisfies the Jacobi identity and the Leibniz rule. This makes $C^1(T^*M, \mathbb{R})$ into a Poisson algebra.

Remark: Hamilton's equations can be rewritten as $\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}$, where $f = q^i, p^i$.

Conserved Quantities

Definition (Conserved Quantity)

Let $f : T^*M \to \mathbb{R}$ be a smooth function, and let $H : T^*M \to \mathbb{R}$ be a Hamiltonian. We say f is a **conserved quantity** if $\{f, H\} = 0$.

Intuitively, this means that the function f is constant along the Hamiltonian flow generated by H.

Example

Let $H = p^2/2m$ be the Hamiltonian of a free particle. Then the momentum p is a conserved quantity (i.e. $\frac{\mathrm{d}p}{\mathrm{d}t} = 0$ or equivalently $\{H, p\} = 0$.)

Next we will connect these to the concept of symmetry.

Symmetry

Symmetries

Definition (Symplectic Group Action)

Let G be a Lie group acting on a manifold M. We say this action is **symplectic** if for each $g \in G$, $(\mathrm{d}L_g)^*\Omega = \Omega$.

Definition (Fundamental Vector Field)

Let G be a Lie group acting on M, let $\mathfrak{g} = T_eG$ be the Lie algebra of G, and let $X \in \mathfrak{g}$. Then the fundamental vector field $X^\# \in \mathfrak{X}(T^*M)$ associated to X is defined by the equation,

$$X^\#|_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp(tX) \cdot (q,p)), \qquad (q,p) \in T^*M \tag{11}$$

Definition (Infinitesimal Symmetry)

Let G be a Lie group acting on M, let $X \in \mathfrak{g}$, and let H be a Hamiltonian. We say that X is an **infinitesimal symmetry** or **symmetry generator** for H if $[X_H, X^\#] = 0$.

Noether's Theorem

Theorem (Noether's Theorem - Hamiltonian Version)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on M, and let $H : T^*M \to \mathbb{R}$ be a Hamiltonian. If each $X \in \mathfrak{g}$ gives an infinitesimal symmetry $X^\#$ of H, then one can construct a function $\mu : T^*M \to \mathfrak{g}^*$, called the **momentum map**, satisfying the differential equation

$$\mathrm{d}\langle \mu(p), X\rangle = X^\# \lrcorner \Omega|_p, \qquad \forall X \in \mathfrak{g} \tag{12}$$

such that each $\mu^X : p \mapsto \langle \mu(p), X\rangle \in \mathbb{R}$ is a conserved quantity.

Remark: Given a Lie group G of dimension k, whose Lie algebra consists of infinitesimal symmetries, we get k independent conserved quantities.
Remark: This gives us a procedure for calculating conserved quantities. Starting with a basis for the Lie algebra $\mathfrak{g}$, we write down their fundamental vector fields, then plug them into the symplectic form, and then integrate the result.

Example: Rotational Symmetry

Example (Rotational Symmetry)

Let $M = \mathbb{R}^2 \setminus \{0\}$ and put Cartesian coordinates (x, y, p_x, p_y) on T^*M, with standard metric $g = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta$. Consider the Kepler Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) - \frac{k}{\sqrt{x^2 + y^2}}$$

We first convert to polar coordinates $(r, \theta, p_r, p_\theta)$. Here, $r = \sqrt{x^2 + y^2}$, and $\theta = \arctan 2(y, x)$. One can compute

$$\partial_r = (x\partial_x + y\partial_y)/\sqrt{x^2 + y^2} \implies p_r = (xp_x + yp_y)/\sqrt{x^2 + y^2}$$
$$\partial_\theta = (x\partial_y - y\partial_x)/\sqrt{x^2 + y^2} \implies p_\theta = (xp_y - yp_x)\sqrt{x^2 + y^2}$$

This yields,

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r}$$

Example: Rotational Symmetry (Cont.)

Example (Rotational Symmetry (Cont.))

We arrived at,

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r}$$

Observe that H has no dependence on θ, so it is invariant under the action of $\mathrm{SO}(2)$ sending $(r, \theta, p_r, p_\theta) \mapsto (r, \theta + \Delta\theta, p_r, p_\theta)$. Setting $\mathfrak{so}(2) = \mathrm{span}(X_1)$, we have $X_1^\# = \partial_\theta$, and one will find that $[X_H, X_1^\#] = 0$. The momentum map equation is,

$$\begin{aligned} \mathrm{d}\mu^1 &= \partial_\theta \lrcorner (\mathrm{d}x \wedge \mathrm{d}p_x + \mathrm{d}y \wedge \mathrm{d}p_y) \\ &= \frac{-y}{\sqrt{x^2+y^2}}\mathrm{d}p_x + \frac{x}{\sqrt{x^2+y^2}}\mathrm{d}p_y \\ &= \mathrm{d}p_\theta \end{aligned}$$

So $\mu^1 = p_\theta + C$ is a conserved quantity for any constant C. In other words, angular momentum is conserved.

Integrability

Definition (Integrable Hamiltonian System)

A Hamiltonian system (M, H, Ω) is said to be **completely integrable** if there exist n independent conserved quantities $Q_1, ..., Q_n$ so that $\{H, Q_i\} = 0$ and $\{Q_i, Q_j\} = 0$ for all i, j.

Remark: The point here is that given k conserved quantities (possibly including H), we can reduce Hamilton's equations from a system of $2n$ ODEs to a system of $2n - k$ ODEs. If $k = n$, we get complete integrability.
Remark: The level sets of each Q_i are invariant Lagrangian submanifolds of T^*M which form a regular foliation (hence the term 'integrable').

Superintegrability

By looking only at actions of a Lie group G on M, we can only reduce the dimension of Hamilton's equations from $2n$ to n. Can we acquire more conserved quantities using a more general method?

Definition (Superintegrable)

A Hamiltonian system is said to be **superintegrable** if there exist $k > n$ independent conserved quantities whose Poisson brackets with each other vanish. If $k = 2n - 1$, we say it is maximally superintegrable.

Theorem

The solutions to a maximally superintegrable system follow closed orbits.

Proof: Suppose we have a superintegrable system. Then there are $2n - 1$ conserved quantities, and the solution to the system lies on the intersection of their level sets. Since the conserved quantities are independent, the dimension of the intersection is 1. With some more work, it can be shown that these orbits close.

As we saw previously, besides H the majority of conserved quantities we seem to find are linear in the momenta p^i. We found conserved linear momentum p_x, p_y as well as conserved angular momentum $xp_y - yp_x$. Are there more conserved quantities which are higher degree polynomials in the momenta?

Isometries

Recall that the Hamiltonian for the Kepler problem was of the form,

$$H = \frac{1}{2m} g^{-1}(p,p) + \frac{k}{f(q)}$$

Where g^{-1} is the induced metric on T^*M. For any group action to be an ordinary symmetry of H, it must therefore preserve the metric. Therefore any symmetry group is a subgroup of the isometry group.

Definition (Killing Vector Field)

Let g be a metric on a manifold M. Recall that the isometry group is the group $\mathrm{Iso}(M,g)$ of diffeomorphisms of M which preserve g. The fundamental vector fields $X^\#$ of $\mathrm{Iso}(M,g)$ are called the **Killing fields**, and obey the **Killing equation**,

$$\nabla_i^g X_j^\# + \nabla_j^g X_i^\# = 0$$

Killing Tensors

Theorem

Let H be a Hamiltonian invariant under $\mathrm{Iso}(M, g)$. Let $f : T^*M \to \mathbb{R}$ be a function of the form $f(q, p) = K_{i_1 i_2 \ldots i_k}(q) p^{i_1} \ldots p^{i_k}$. Then for f to be conserved it is necessary that,

- The coefficients K_I, for each $|I|$, form the components of a symmetric tensor field.
- Each K_I satisfies the generalized Killing equation,

$$\mathrm{Sym}(\nabla^g K) = 0 \tag{13}$$

Such a K is called a Killing tensor field.

Remark: Conserved quantities arising from a Killing tensor of rank ≥ 2 can not always be derived from a Lie group action on M. Sometimes, you can get them by acting on T^*M.
Remark: It is often easier to find these by solving $\{H, f\} = 0$ rather than $\mathrm{Sym}(\nabla^g K) = 0$.

Example: Laplace-Runge-Lenz Vector

Example (Laplace-Runge-Lenz Vector)

Continuing with our example of the Kepler problem in $\mathbb{R}^2$, we have conserved quantities,

$$H = \frac{1}{2m} p_r^2 + \frac{p_\theta^2}{2m} \frac{1}{r^2} - \frac{k}{r}, \qquad Q^1 = p_\theta$$

Let us look for others of the form

$$K_1 p_r^2 + K_2 p_\theta^2 + K_3 p_r p_\theta + W$$

We first plug this into the equation $\{H, f\} = 0$ and expand in like powers of p_r, p_θ. We arrive at a system of DEs with two linearly independent solutions,

$$A_1 = -r p_r p_\theta \sin\theta + (r p_\theta^2 - mk)\cos\theta \tag{14}$$

$$A_2 = r p_r p_\theta \cos\theta + (r p_\theta^2 - mk)\sin\theta \tag{15}$$

The vector $\mathrm{A} = (A_1, A_2)$ is called the Laplace-Runge-Lenz vector.

Example: Laplace-Runge-Lenz Vector (Cont.)

Example (Laplace-Runge-Lenz Vector (Cont.))

Now we have the conserved quantities

$$A_1 = -rp_r p_\theta \sin\theta + (rp_\theta^2 - mk)\cos\theta \tag{16}$$

$$A_2 = rp_r p_\theta \cos\theta + (rp_\theta^2 - mk)\sin\theta \tag{17}$$

These are linearly independent, but are they independent in the sense that $\{A_1, Q_i\} = 0$? No, we have

$$\{p_\theta, A_1\} = -A_2, \quad \{p_\theta, A_2\} = -A_1$$
$$\{A_1, A_2\} = -2mHp_\theta$$

Furthermore,

$$A_1^2 + A_2^2 = m^2k^2 + 2mHp_\theta^2$$

So the magnitude of A is not independent of the other quantities. However, the **direction**, $\arctan(A_2/A_1)$ is.

Acknowledgements

- Thank you to my supervisors, Shengda Hu and Manuele Santoprete (Wilfrid Laurier University)