UNIVERSITY OF MANITOBA

DATE: March 17, 2015

TERM TEST 2
TITLE PAGE

EXAMINATION: Engineering Mathematical Analysis 2
COURSE: MATH 2132

TIME: 70 minutes
EXAMINER: A. Clay

FAMILY NAME: (Print) Solutions

GIVEN NAME(S):

STUDENT NUMBER:

SIGNATURE:
(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 70 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 5 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 50 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

[10] 1. Solve the initial value problem

$$x^2y' + xy = 1 + xe^x, \ x > 0 \text{ and } y(1) = 2.$$

Divide by x^2:

$$y' + \frac{1}{x}y = \frac{1}{x^2} + \frac{1}{x}e^x, \quad x > 0.$$

Now it's first-order linear with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{1}{x}e^x$.

$$\boxed{\text{So } \mu(x) = e^{\int P(x)dx}} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x \quad \left(\text{note abs. value dropped as } x>0\right)$$

Then
$$\int \mu(x)Q(x)dx = \int x\left(\frac{1}{x^2} + \frac{1}{x}e^x\right)dx$$

$$= \int \frac{1}{x} + e^x \, dx = \ln|x| + e^x$$

And solution is

$$y(x) = \frac{1}{x}\left(\ln|x| + e^x + C\right)$$

Then $y(1) = 2$

$$\Rightarrow 2 = \frac{1}{1}(\cancel{\ln(1)}^{0} + e + C) \Rightarrow C = 2 - e$$

Solution is

$$y(x) = \frac{1}{x}\left(\underline{\ln|x|} + e^x + 2 - e\right)$$

no abs needed, $x > 0$.

[10] 2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy^2}{1+x^2}, \; y(0) = 1.$$

This is separable. So

$$\frac{dy}{dx} = \left(\frac{x}{1+x^2}\right) y^2$$

$$\Rightarrow \int \frac{1}{y^2}\, dy = \int \frac{x}{1+x^2}\, dx$$

$$\frac{-1}{y} = \frac{1}{2} \ln|1+x^2| + C$$ (can drop abs values since $1+x^2>0$)

$$\Rightarrow y = \frac{-1}{\frac{1}{2}\ln(1+x^2) + C}$$

Use $y(0)=1$:

$$1 = \frac{-1}{\frac{1}{2}\ln(1+0^2) + C} \Rightarrow 1 = \frac{-1}{C}$$

$$\Rightarrow C = -1.$$

So

$$y = \frac{-1}{\frac{1}{2}\ln(1+x^2) - 1}.$$

[10] 3. Find a two-parameter family of solutions to

$$xy'' - 2y' = 2x^3$$

Here, y is missing so we substitute $v = y'$, $v' = y''$ and get

$$xv' - 2v = 2x^3,$$ first order linear.

$$\Rightarrow v' - \frac{2}{x}v = 2x^2$$

So $P(x) = \frac{-2}{x}$, $Q(x) = 2x^2$ so

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{-2}{x}\,dx} = e^{-2\ln|x|} = e^{\ln(x^{-2})} = \boxed{\frac{1}{x^2}}$$

So $\int Q(x)\mu(x) = \int 2x^2 \cdot \frac{1}{x^2}\,dx = 2x.$

So a solution for v is

$$v = \frac{2x + C}{1/x^2} = 2x^3 + Cx^2$$

Then $y = \int v = \int 2x^3 + Cx^2\,dx = \frac{1}{2}x^4 + \frac{C}{3}x^3 + D$

$$= \frac{1}{2}x^4 + Cx^3 + D$$

new arbitrary constant.

[7] 4. (a) What does it mean for functions to be linearly independent? Show that $y_1(x) = x$, $y_2(x) = x^2$ and $y_3(x) = e^{\pi x}$ are linearly independent.

Functions $y_1(x), y_2(x), \dots, y_n(x)$ are linearly independent if $c_1y_1 + c_2y_2 + c_3y_3 + \dots + c_ny_n = 0$ implies $c_1 = c_2 = \dots = c_n = 0$.

Here, $c_1y_1 + c_2y_2 + c_3y_3 = 0$

$$\Rightarrow c_1 x + c_2 x^2 + c_3 e^{\pi x} = 0$$

Set $x=0$. Then we get $0+0+c_3=0$, so $c_3=0$.

Then we have $c_1 x + c_2 x^2 = 0$. Set $x=1$, and -1. We get respectively $c_1 + c_2 = 0$ ① and $-c_1 + c_2 = 0$ ②

$\Rightarrow c_1 = c_2$ sub into ①.

So $c_2 + c_2 = 0$

$\Rightarrow c_2 = 0$ and so $c_1 = 0$.

Therefore the functions are independent.

[3] (b) Suppose that a third-order linear homogeneous differential equation has $y_1(x) = x$, $y_2(x) = x^2$ and $y_3(x) = e^{\pi x}$ as solutions. Is

$$y(x) = c_1y_1(x) + c_2y_2(x) + c_3(y_1(x) + y_2(x))$$

a general solution? Justify your answer (Hint: Use part (a)).

No, it is not a general solution. If it were then we'd have

$$e^{\pi x} = c_1 x + c_2 x^2 + c_3(x + x^2)$$

$$= (c_1 + c_3)x + (c_2 + c_3)x^2$$

But this is not possible by part (a).

[10] 5. Find the general solution to

$$y'' - 2y' - 8y = e^{-2t} + 1$$

The complementary equation is

$$m^2 - 2m - 8 = 0$$

with roots found by factoring: $(m-4)(m+2) = 0$

roots: $4, -2$.

Therefore $y_h(t) = c_1 e^{4t} + c_2 e^{-2t}$.

Using undetermined coefficients we guess

$$y_p(t) = Ate^{-2t} + B.$$

Then $y_p' = Ae^{-2t} - 2Ate^{-2t}$.

$$y_p'' = -2Ae^{-2t} - 2(Ae^{-2t} - 2Ate^{-2t})$$
$$= -4Ae^{-2t} + 4Ate^{-2t}.$$

Plugging in we get:

$$-4Ae^{-2t} + \cancel{4Ate^{-2t}} - 2(Ae^{-2t} - \cancel{2Ate^{-2t}}) - 8(\cancel{Ate^{-2t}} + B) = e^{-2t} + 1.$$

We get

$$-4A - 2A = 1 \quad (\text{coefficients of } e^{-2t})$$
$$-8B = 1 \quad (\text{coeffs of constant})$$

$$\Rightarrow A = \frac{-1}{6} \quad B = \frac{-1}{8}$$

So $y(x) = c_1 e^{4t} + c_2 e^{-2t} + \left(\frac{-1}{6}\right) te^{-2t} - \frac{1}{8}$

is the general solution.