UNIVERSITY OF MANITOBA

DATE: February 10, 2015 TERM TEST 1

TITLE PAGE

EXAMINATION: Engineering Mathematical Analysis 2 TIME: 70 minutes

COURSE: MATH 2132 EXAMINER: A. Clay

FAMILY NAME: (Print) Solutions.

GIVEN NAME(S):

STUDENT NUMBER:

SIGNATURE:

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 70 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 4 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 50 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

[10] 1. Find the limit of the sequence of functions

$$f_n(x) = \tan^{-1}(nx) + \frac{nx+2}{2n-1}$$

There are 3 cases:

1) $x > 0$.

Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1}(nx) + \lim_{n\to\infty} \frac{nx+2}{2n-1}$$

$$= \frac{\pi}{2} + \lim_{n\to\infty} \frac{x+2/n}{2-1/n}$$

$$= \frac{\pi}{2} + \frac{x}{2}.$$

2) $x = 0$

Then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1}(0) + \lim_{n\to\infty} \frac{2}{2n-1}$

$$= 0 + 0 = 0.$$

3) $x < 0$

Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1}(nx) + \lim_{n\to\infty} \frac{nx+2}{2n-1}$$

$$= -\frac{\pi}{2} + \frac{x}{2}.$$

So the limit is

$$f(x) = \begin{cases} \frac{\pi}{2} + \frac{x}{2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\frac{\pi}{2} + \frac{x}{2} & \text{if } x < 0 \end{cases}$$

[4] 2. (a) Calculate the first three Taylor polynomials $P_0(x), P_1(x), P_2(x)$ of the function $f(x) = \cos(x) + \sin(x)$ about 0.

can either ① Take the first few terms of the Taylor series formula, or ② Calculate derivatives and find the polynomials from scratch.

$$P_0(x) = \cos(0) + \sin(0) = 1$$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x - \frac{x^2}{2}$$

[6] (b) Use Taylor's remainder formula to show that the MacLaurin series of the function $f(x) = \cos(x) + \sin(x)$ about 0 converges to $f(x)$ for all x.

The remainder formula is

plug in 0 for 'c'

$$R_n(0, x) = R_n = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-0)^{n+1}$$

Here, the derivative $f^{(n+1)}(z_n)$ is always of the form $\pm\cos(x) \pm \sin(x)$ or $\pm\cos(x) \mp \sin(x)$. Either way we get

$$-2 \le f^{(n+1)}(z_n) \le 2 \quad \text{for all } z_n$$

$$\Rightarrow \quad \frac{-2x^{n+1}}{(n+1)!} \le \frac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1} \le \frac{2x^{n+1}}{(n+1)!}$$

So taking limits gives

$$0 \le \lim_{n\to\infty} \frac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1} \le 0$$

So $\displaystyle\lim_{n\to\infty} \frac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1} = 0$, by the squeeze theorem

[10] 3. Find the open interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{(2n+3)}(x-1)^{2n}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(-1)^n e^n}{2n+3}}{\frac{(-1)^{n+1} e^{n+1}}{2n+5}} \right| = \lim_{n\to\infty} \left| \frac{e^n (2n+5)}{(2n+3)e^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{2n+5}{2n+3} = \frac{1}{e}$$

So we have convergence for

$$-\frac{1}{e} < (x-1)^2 < \frac{1}{e}$$

$$\Rightarrow \frac{-1}{\sqrt{e}} < x-1 < \frac{1}{\sqrt{e}}$$

$$\Rightarrow 1 - \frac{1}{\sqrt{e}} < x < 1 + \frac{1}{\sqrt{e}}$$

[10] 4. Find the power series for the function

$$f(x) = \frac{1}{3x-2} + \ln(x)$$

centered at 1 and state your answer as a single sum. State the open interval of convergence as well.

Here, $f(x)$ is the sum of two functions to be handled separately.

1) From class they know

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \text{ for } 0 < x < 2$$

2) Set $y = x-1$. Then $x = y+1$ and

$$\frac{1}{3x-2} = \frac{1}{3(y+1)-2} = \frac{1}{3y+1} = \frac{1}{1-(-3y)}$$

The geometric series formula $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$, $-1<x<1$

gives

$$\frac{1}{1-(-3y)} = \sum_{n=0}^{\infty} (-3)^n y^n \text{ for } -1 < 3y < 1 \Rightarrow -\frac{1}{3} < y < \frac{1}{3}$$

$$\Rightarrow \boxed{\frac{1}{3x-2} = \sum_{n=0}^{\infty} (-3)^n (x-1)^n} \text{ for } \boxed{\frac{2}{3} < x < \frac{4}{3}.}$$

Then as one sum, we get

$$1 + \sum_{n=1}^{\infty}\left((-3)^n + \frac{(-1)^{n+1}}{n}\right)(x-1)^n \text{ with}$$

convergence on the smaller of the two intervals: $\frac{2}{3} < x < \frac{4}{3}$

[10] 5. Find the sum of the power series

$$\sum_{n=1}^{\infty}(n+1)x^{n-1}$$

State its open interval of convergence as well.

There may be multiple ways of doing this question!

Set $S(x) = \sum_{n=1}^{\infty}(n+1)x^{n-1}$.

Integrate twice:

$$\iint S(x) = \sum_{n=1}^{\infty}\frac{\cancel{(n+1)}}{n\cancel{(n+1)}}x^{n+1} = \sum_{n=1}^{\infty}\frac{1}{n}x^{n+1} + Cx + D$$

Then factor out an x:

$$\iint S(x) = x\sum_{n=1}^{\infty}\frac{1}{n}x^{n} + Cx + D$$

But $\ln(1-x)$ has formula:

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{1}{n}x^{n} \quad \text{for } -1<x<1$$

So $$\iint S(x) = -x\ln(1-x) + Cx + D$$

Now diff. twice:

$$\int S(x) = \frac{x}{1-x} - \ln(1-x) + C$$

$$S(x) = \frac{1}{1-x} + \frac{+x}{(1-x)^2} + \frac{1}{1-x}$$

$$= \frac{2(1-x)+x}{(1-x)^2} = \frac{2-x}{(1-x)^2} \quad -1<x<1$$