An algorithm for determining non-bi-orderability of knot groups.

Colin Desmarais

A group G is bi-orderable if there exists an ordering $<$ of the elements of G that is invariant under left and right multiplication:

$$g<h \Rightarrow fg<fh \text{ and } gf<hf \quad \forall f,g,h \in G.$$

For a knot K, we'll say "K is bi-orderable" if $\pi_1(S^3 \setminus K)$ is bi-orderable.

For a knot K, let $\Delta_K(t)$ denote the Alexander polynomial. We can use $\Delta_K(t)$ to determine which knots have bi-orderable group (sometimes).

Example: K is the trefoil knot:

Its group is $\langle x, y \mid x^3 = y^2 \rangle$, and we compare xy and yx. If

$xy < yx$ then $x^2xy < x^2yx \Rightarrow y^3 < x^2yx$

and $\Rightarrow xyx^2 < yx^3 = y^3$.

So $xyx^2 < y^3 < x^2yx$

Conjugate $x^{-1}(\ )x \Rightarrow yx < xy$, a contradiction.

So, K is not-bi-orderable.

What else is known about bi-orderability of knot groups?

Theorem. (Clay-Rolfsen) (Rolfsen-Perron).

Let K be a fibred knot.

(1) If all the roots of $\Delta_K(t)$ are positive reals, then K is bi-orderable.

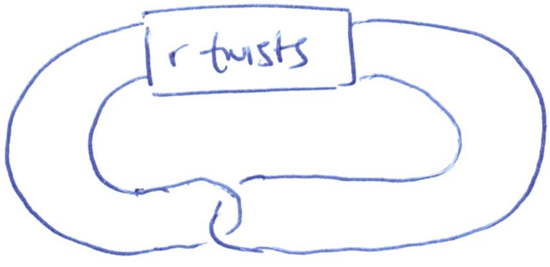
(2) If none of the roots of $\Delta_K(t)$ are positive reals, then K is not bi-orderable.

Theorem (Clay, D, Naylor).

Let K be a two ~~d~~ bridge knot. If none of the roots of $\Delta_K(t)$ are real and positive, then K is not bi-orderable.

There are also some additional special cases, for example 9_{16} is 3-bridge but it's non bi orderable.

Additionally, for twist knots K_r of the form:



We have

<u>Theorem</u>: K_r is a twist knot, then
(i) If r is even, then K_r is bi-orderable
(ii) If r is odd, then K_r is not bi-orderable.

This seems to look like $\Delta_K(t)$ plays a significant role in knot bi-orderability. Unfortunately,

<u>Theorem</u>: For any knot K there exists another knot K' with $\Delta_K(t) = \Delta_{K'}(t)$, with K' non-bi-orderable.

However, there is hope that:

<u>Question</u>: If $\Delta_K(t)$ has no positive real roots, does this mean K is <u>non-bi-orderable</u>?

For fewer than 10 crossings, the knots having $\Delta_K(t)$ with no positive real roots that <u>might</u> be non-bi-orderable (we don't know) are

8_{15}, 9_{35}, 9_{38}, 9_{41}, 9_{49}.

All other knots with fewer than 10 crossings and $\Delta_K(t)$ has not real roots is known to be non-bi-orderable.

<u>Proposition</u>: If a group G is bi-orderable, there exists $P \subset G$ so that

(i) $P \cdot P \subset P$

(ii) $P \cup P^{-1} = G \setminus \{1\}$

(iii) $P \cap P^{-1} = \emptyset$

(iv) $gPg^{-1} \subset P$.

This is called the <u>positive cone</u> of the ordering, the ordering is defined by $g < h \iff g^{-1}h \in P$.

<u>Proposition</u>: Suppose H is a normal subgroup of G, and set $L = G/H \cong \mathbb{Z}$. Consider the short exact sequence

$$1 \longrightarrow H \hookrightarrow G \longrightarrow \underset{\cong \mathbb{Z}}{L} \longrightarrow 1$$

Then G is bi-orderable if and only if both H and L are bi-orderable and the ordering on H is <u>invariant under conjugation by elements of G</u>.

Together these propositions say that

if G is bi-orderable and $H \trianglelefteq G$ then there exists $P_H \subset H$ so that

- $P_H \cdot P_H \subset P_H$
- $P_H \cup P_H^{-1} = H \setminus \{1\}$
- $P_H \cap P_H^{-1} = \emptyset$
- $g P_H g^{-1} \subset P_H$ for all $g \in G$.

Fix a set S of generators of G. Then set

$H_n =$ elements represented by words of length n in the gen set S $\qquad$ $G_m =$ elements of G rep. by words of length m in the gen set S.

So, if G is bi-orderable then $\exists\, Q_{n,m} \subset H$ with

(i) $(Q_{n,m} \cdot Q_{n,m}) \cap H_n \subset Q_{n,m}$

(ii) $Q_{n,m} \cup Q_{n,m}^{-1} = H_n \setminus \{1\}$

(iii) $Q_{n,m} \cap Q_{n,m} = \emptyset$

(iv) $(g Q_{n,m} g^{-1}) \cap H_n \subset Q_{n,m}$ for all $g \in G_m$.

See the output below for S_{15}