The space of left-orderings II

The goal of this lecture is to solve a longstanding open problem using the tools introduced last time.

Recap:

We identified every left-invariant ordering of a group G with its positive cone

$$P = \{g \in G \mid g > 1\}.$$

The set of such cones is a closed subset of $\mathcal{P}(G)$, we call it $LO(G)$:

$$LO(G) \subset \mathcal{P}(G) = \prod_{g \in G} \{0,1\}$$

and so it inherits a topology from the product, making it into a compact space. In fact, a subbasis was

$$U_g = \{P \in LO(G) \mid g \in P\}$$

$$\text{and } U_g^c = U_{g^{-1}} = \{P \in LO(G) \mid g^{-1} \in P\}.$$

This made $LO(G)$ Hausdorff, totally disconnected and metrizable if G is countable.

We'll use these facts to prove:

Theorem (Linnell 2007)

If a group G is left orderable, then it has either uncountably many orderings or 2^n orderings for some n.

This answers a long-standing problem, which asked if it were possible for a group to have countably infinitely many left-orderings (Open for ~40 years).

Previous work had been able to deal with the following case: (Late 80's)

Theorem: If G has a left-ordering which restricts to a bi-ordering on some finite index subgroup H of G, then G has uncountably many left-orderings.

To solve the problem (ie. prove Linnell's theorem), let's make a few basic topological observations:

① Let G be a group and X a compact, Hausdorff topological space, and suppose G acts on X by homeomorphisms. Then X contains a minimal invariant set $M \subset X$, ie a compact set satisfying:

(i) $g(M) = M \;\; \forall g \in G$, and

(ii) M contains no proper invariant subset.

Proof: Set

$$S = \{A \subset X \mid A \text{ is nonempty, compact and } G\text{-invariant}\}$$

Then S is nonempty since it contains X; and ordered by inclusion. Moreover, any chain of compact, nonempty G-invariant sets

$$A_1 \supset A_2 \supset A_3 \supset A_4 \supset \dots$$

will have the finite intersection property, so by compactness of X

$$\bigcap_{i \in I} A_i \neq \emptyset,$$ and the intersection is compact & G-invariant.

So, every chain in S has a lower bound.

Zorn's Lemma $\Rightarrow$ M exists.

Remark: These sets are well studied, and have many appealing properties, but we need only one:
Given $x \in M$, let $G(x)$ denote the orbit of x under the G-action. Then $\overline{G(x)}$ is a closed, hence compact, G-invariant subset. Since $x \in M$, $\overline{G(x)} \subset M$. But M is minimal so in fact $\overline{G(x)} = M$.

(2) A compact Hausdorff space with no one-point open sets is uncountable.

Proof: Suppose X is countable, compact and Hausdorff. Say $X = \{x_i\}_{i=1}^{\infty}$.

Then each set $\{x_i\}$ is closed and has empty interior, since no singleton is open. Then

$$X = \bigcup_{i=1}^{\infty} \{x_i\}$$

expresses X as a countable ~~collection~~ union of closed sets with empty interior, contradicting Baire category.
So X is uncountable.

(3) A left-orderable group G acts on its corresponding space ~~by~~ $LO(G)$ by homeomorphisms.

The action of $g \in G$ on $P \in LO(G)$ is

$$g(P) = gPg^{-1},$$

the key is to note that the subset $gPg^{-1} \subset G$ satisfies the two properties of a positive cone:

(i) $gPg^{-1} \cdot gPg^{-1} \subset gPg^{-1}$

(ii) $gPg^{-1} \cup gP^{-1}g^{-1} \cup \{1\} = G.$

Moreover, the action is by homeomorphisms since it actually sends sub-basic open sets to sub-basic open sets:

$$U_h = \{P \in LO(G) \mid h \in P\}$$

then

$$\begin{aligned} g(U_h) &= \{gPg^{-1} \in LO(G) \mid h \in P\} \\ &= \{P \in LO(G) \mid g^{-1}hg \in P\} \\ &= U_{g^{-1}hg}. \end{aligned}$$

So the action is indeed by homeomorphisms.

Proof that $LO(G)$ is either finite or uncountable:

Since G acts on $LO(G)$ by homeomorphisms, the action has a minimal invariant set, $M \subset LO(G)$. There are two cases:

① M is infinite. If M has no open singletons, then M is uncountable and we're done (by ②). So suppose $P \in M$ is a positive cone that is an isolated point (in M). Then $\overline{G(P)} = M$ and so $G(P)$ is infinite (since M is infinite) and thus $G(P)$ has an accumulation point $Q \in M$, as M is compact. But then

(i) $Q \notin G(P)$ since Q is an accumulation point and all point in $G(P)$ are isolated,

(ii) $\overline{G(Q)} = M \supset P$, meaning P cannot be isolated since it's in $\overline{G(Q)}$. Contradiction.

Thus M has no isolated points, is compact & Hausdorff.
$\Rightarrow$ uncountable

② If M is finite, then choose $P \in M$ and observe that $G(P)$ is finite. So the subgroup

$$G_P = \{ g \in G \mid gPg^{-1} = P \}$$

is finite index. But then the subset

$P \cap G_p \subset G_p$ satisfies all of the conditions to be the positive cone of a <u>bi-ordering</u> of G_p.

$\Rightarrow$ $LO(G)$ is uncountable, by earlier work.

What other things have been proved this way, using $LO(G)$ and some sort of compactness trick?

<u>Thm</u>: If every finitely generated subgroup of G is LO, then G is LO

(Applic'n of the finite intersection property)

<u>Thm</u> (Burns-Hale)

If every f.g subgroup H of G admits a surjection onto a nontrivial LO grp, then G is LO.

<u>Thm</u>: If G is amenable and LO, then ...

(i) Every f.g. subgroup of G surjects onto $\mathbb{Z}$

(ii) $\exists$ a special kind of LO of G,

... etc.