Introduction to the space of orderings I

Let G be a group. Then G is left-orderable if there exists a strict total ordering $<$ of G such that $g<h \Rightarrow fg<fh \ \forall f,g,h \in G$.

A left-ordering becomes a bi-ordering if it also satisfies $g<h \Rightarrow gf<hf \ \forall f,g,h \in G$.

Each left-ordering of G is associated to a unique positive cone $P \subset G$, satisfying

(i) $P \cdot P \subset G$

(ii) $P \cup P^{-1} \cup \{1\} = G$ (disjoint union).

The bijection between cones and orderings is given by:

If $<$ is a LO of G, define P by

$$P = \{g \in G \mid g > 1\}$$

and conversely, given P we define $<$ according to

$$g<h \Leftrightarrow g^{-1}h \in G.$$

For bi-orderings, we also require P to satisfy:

iii) $gPg^{-1} \subset P \ \forall g \in G$.

Examples of LO groups which are not BO :

- The group $\langle x, y \mid xyx^{-1} = y^{-1} \rangle$ is LO but not BO. It's LO because $1 \longrightarrow \mathbb{Z} \longrightarrow \langle x, y \mid xyx^{-1} = y^{-1} \rangle \longrightarrow \mathbb{Z} \longrightarrow 1$, and the middle of a short exact sequence can be lex ordered. It's not BO because y is conj. to its inverse, so if $y > 1 \Rightarrow xy > x \Rightarrow \underset{\underset{y^{-1}.}{\parallel}}{xyx^{-1}} > 1$, contradiction.

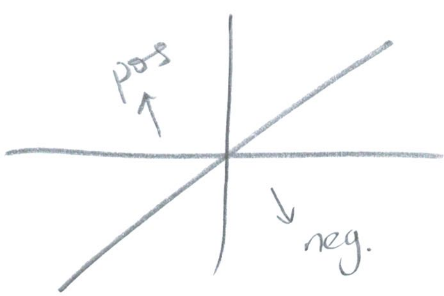
- The group $B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j|=1 \end{array} \right\rangle$.

To see it's LO is tough, but not BO comes from $\sigma_1\sigma_2^{-1}$ being conjugate to its inverse:

$$(\sigma_1\sigma_2\sigma_1)(\sigma_1\sigma_2^{-1})(\sigma_1\sigma_2\sigma_1)^{-1} = \sigma_2\sigma_1^{-1}.$$

Examples of BO groups :

- $\mathbb{Z}^n$, e.g. in $\mathbb{Z}^2$ we make an ordering by choosing a line L of irrational slope and declaring one side positive :



- F_n, the free group on n generators (Magnus order)

• P_n, the pure braids of B_n (they are the kernel of the "follow the strands" homomorphism onto S_n.

$$1 \to P_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

Now given a group G, we define two spaces $BO(G) \subset LO(G)$ as follows:

Let $P(G)$ denote the power set of G. Then

$P(G) = \prod_{g \in G} \{0,1\}$ as a set, where a subset

$S \in P(G)$ is identified with $s \in \prod_{g \in G} \{0,1\}$ satisfying

$p_g(s) = 1$ if and only if $g \in S$. (I.e, the projection map in the g^{th} component maps the element to 1 iff $g \in S$).

Therefore $P(G)$ has a natural topology, namely the product topology on $\prod_{g \in G} \{0,1\}$. A subbasis for the topology is all the sets

$$V_g = \{S \subset G \mid g \in S\}$$

$$V_g^c = \{S \subset G \mid g \notin S\}.$$

Now set

$$BO(G) = \{P \subset G \mid P \text{ is the positive cone of a bi-ordering}\}$$

and

$$LO(G) = \{P \subset G \mid P \text{ is the positive cone of a left-ordering}\}.$$

Then

$$BO(G) \subset LO(G) \subset \mathcal{P}(G),$$

and so both inherit a subspace topology, whose subbasis is

$$U_g = \{P \subset G \mid g \in P\}$$

and $\underbrace{U_g^c = U_{g^{-1}}}_{\text{because } P \cup P^{-1} \cup \{1\} = G.} = \{P \subset G \mid g^{-1} \in P\}$

Recall: Totally disc. means that every subset S with $|S| > 1$ is disconnected

<u>Theorem</u>: $BO(G)$ and $LO(G)$ are totally disconnected, Hausdorff and compact. If G is countable, they are metrizable.

<u>Proof</u>: Aside from compactness, all of these properties follow from being a subset of $\mathcal{P}(G)$ with the subspace topology.

E.g. $\mathcal{P}(G)$ is Hausdorff because if S, T are distinct subsets of G then $\exists\, g \in T \setminus S$ or $g \in S \setminus T$. In the first case,

$$T \in U_g \text{ and } S \in U_g^c, \text{ and } U_g \cap U_g^c = \emptyset$$

while in the second,

$$S \in U_g \text{ and } T \in U_g^c.$$

Note this also shows that sets of the form $\{U_g, U_g^c\}$

can be used as a separation of any subset of $P(G)$ containing more than one element (hence totally disc).

For compactness, we need only show that $BO(G)$ and $LO(G)$ are <u>closed</u> subsets. To do this, we show the complement is open, by showing that "failure to be a pos cone" is an open condition.

E.g. If $S \subset G$ does not satisfy $S \cdot S \subset S$, then $\exists\, g, h \in G$, s.t.

$$S \in U_g \cap U_h \cap U_{gh}^c$$

i.e. $S \in \bigcup_{g,h \in G} (U_g \cap U_h \cap U_{gh}^c)$, which is an open set.

Similarly for $P \cup P \cup \{1\} = G$, and $gPg^{-1} \subseteq P$.

<u>Example</u>: If $G = \mathbb{Z}^2$, then in fact <u>all</u> orderings of G arise from lines in $\mathbb{R}^2$, as discussed earlier. There is one technicality that causes a hiccup: If the line has rational slope, then it give 4 orderings by picking a direction along the line to be positive, as well as a side to be positive.

Then let $S_{[\,)}$ and $S_{(\,]}$ denote two copies of S^1 with topologies inherited from the Sorgenfrey line with open sets $[\ )$ and $(\ ]$ respectively.

Theorem: $BO(\mathbb{Z}^2) = LO(\mathbb{Z})^2 \simeq S^1_{(]} \cup S^1_{[)} / \sim$, where $\sim$ identifies corresponding irrational points in the two copies of S^1 (A point $x \in S^1$ is irrational if $x = e^{2\pi i \theta}$ for $\theta \in (0, 1]$ irrational).

In general, determining the structure of $BO(G)$ or $LO(G)$ as above is quite difficult. However, we have:

Theorem: Let X be a nonempty, totally disconnected metric space with no open singletons. Then X is homeomorphic to the Cantor set.

Cor: If G is countable, then $LO(G)$ and $BO(G)$ are homeomorphic to the Cantor set as long as they contain no isolated points.

What is an "isolated point" in $LO(G)$?
Suppose $P \in LO(G)$ and $\{P\}$ is open. Then
$\{P\}$ open $\Rightarrow \exists$ a basic open nbhd U with

$$U = \{P\}.$$

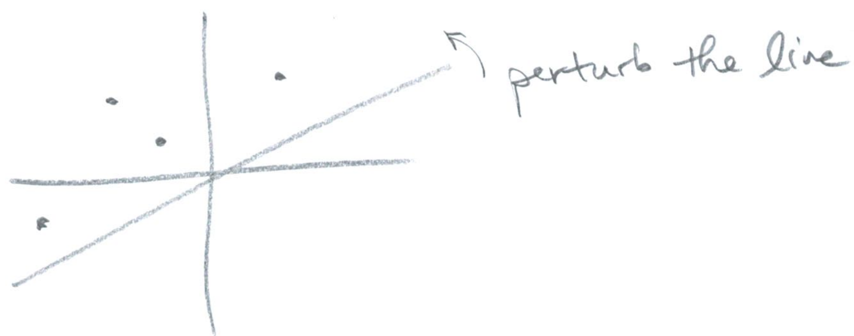
$$\Rightarrow \exists\ g_1, \dots, g_k \text{ such that}$$

$$U_{g_1} \cap \dots \cap U_{g_k} = \{P\}$$

$\Rightarrow P$ is the unique positive cone in $LO(G)$ containing $g_1, \ldots, g_k$.

I.e an isolated point in $LO(G)$ is the unique ordering making some finite collection of elements positive.

<u>Example</u>: $LO(\mathbb{Z}^2)$ has no isolated points:



<u>Example</u>: $LO(F_n)$ and $LO(B_\infty)$ have no isolated points.

<u>Example</u>: $LO(G*H)$ has no isolated pts.

<u>Example</u>: $LO(B_n)$ has isolated points. So does $LO(G_{p,q})$, where

$$G_{p,q} = \langle x, y \mid x^p = y^q \rangle.$$

<u>Question</u>: Does $BO(F_n)$ have isolated points?

Example: Here is a description of the isolated point in B_3.

Recall $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$.

Let $Q \subset B_3$ be the semigroup generated by the elements $\{\sigma_2^{-1}, \sigma_1\sigma_2\}$.

Theorem: $Q \cup Q^{-1} = B_3 \setminus \{1\}$ and
$$Q \cap Q^{-1} = \emptyset,$$
and therefore Q is a positive cone.

Claim: Q is the only positive cone containing σ_2^{-1} and $\sigma_1\sigma_2$.

Proof: Suppose not, say $\sigma_2^{-1}, \sigma_1\sigma_2 \in P \subset B_3$.
Then since $P \cdot P \subset P$, we know $Q \subset P$.
On the other hand, suppose $P \not\subset Q$. Choose $\beta \in P \setminus Q$. Then $\beta^{-1} \in Q$ since $Q \cup Q^{-1} \cup \{1\} = B_3$, and so $\beta, \beta^{-1} \in P \Rightarrow 1 \in P$, a contradiction.
Thus $P \subset Q$, so $Q = P$.