MATH 3472

§ 11.10

Recall that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is <u>periodic</u> of period p if $f(x+p) = f(x)$ $\forall x \in \mathbb{R}$. Set

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \begin{cases} \dfrac{\sin(n+\frac{1}{2})t}{2\sin(t/2)} & \text{if } t \neq 2m\pi \\ & (m \text{ an int}). \\ n + \frac{1}{2} & \text{if } t = 2m\pi \end{cases}$$

The formula $D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \dfrac{\sin(n+\frac{1}{2})t}{2\sin(t/2)}$, $t \neq 2m\pi$

is non-obvious. From

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = f(x)$$

$$\Rightarrow \quad \sin\left(\frac{x}{2}\right) + \sum_{k=1}^{n} 2\sin(x/2)\cos kx = 2\sin\left(\frac{x}{2}\right)f(x)$$

Now use $\cos(a)\sin(b) = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right)$

$$= \sin\left(\frac{x}{2}\right) + \sum_{k=1}^{n}\left(\sin\left(kx + \frac{x}{2}\right) - \sin\left(kx - \frac{x}{2}\right)\right)$$

$$= \sin\left(\frac{x}{2}\right) + \underbrace{\sum_{k=1}^{n}\left[\sin\left(\left(k+\tfrac{1}{2}\right)x\right) - \sin\left(\left(k-\tfrac{1}{2}\right)x\right)\right]}_{\text{telescoping}}$$

$$= \sin\left(\left(n+\tfrac{1}{2}\right)x\right)$$

$$\Rightarrow \quad f(x) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\sin(x/2)}$$

The function $D_n(t)$ is called Dirichlet's kernel.

<u>Theorem</u>: Assume that $f \in L([0,2\pi])$ and suppose f is periodic with period 2π. Let $\{S_n\}$ denote the partial sums

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)).$$

Then $$S_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} D_n(t)\, dt.$$

<u>Remark</u>: Finally we see the purpose of the previous corollaries and of the investigation of the Dirichlet kernel $D_n(t)$.:

The Fourier series $\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$ will converges at a point $x = x_0$ if and only if the limit

$$\lim_{n\to\infty} \frac{2}{\pi} \int_0^{\pi} \frac{f(x_0+t) + f(x_0-t)}{2} D_n(t)\, dt$$

exists.

<u>Proof</u>: Recall that the Fourier coefficients are

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\cos(nt)\, dt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\sin(nt)\, dt,$$

so if we substitute these into the Fourier series, we get (upon rewriting the integrals as a single integral)

$$S_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \left[\cos kt \cos kx + \sin kt \sin kx \right] \right] dt$$

However recall that

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$$

therefore

$$\cos kt \cos kx + \sin kt \sin kx = \cos(k(t-x))$$

So

$$S_n(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\left[\cos(k(t-x))\right]\right] dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\, D_n(t-x)\, dt.$$

By assumption, f (and $D_n(t-x)$) are both periodic of period 2π. Thus we can integrate over $[x-\pi, x+\pi]$ for any x and get the same result.

<u>Aside</u>: If $f(x)$ is ~~der~~ periodic with period p, then $\int_0^p f(x)\,dx = \int_t^{t+p} f(x)\,dx \;\forall t \in \mathbb{R}$. To see this, set $H(t) = \int_t^{t+p} f(x)\,dx$. Then taking derivatives, $H'(t) = f(t+p) - f(t) = 0$, so H is constant.

$$\Rightarrow S_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)\, D_n(t-x)\,dt \quad (\text{set } u = t-x)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\, D_n(u)\, du.$$

But observe that $D_n(-u) = D_n(u)$ (it's a quotient of sines, both odd)

Thus

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\, D(u)\, du$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{f(x+u)+f(x-u)}{2}\, D_n(u)\, du.$$

Thus we want to investigate the convergence of

$$\lim_{n\to\infty} \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2}\; \frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}\, dt$$

This is <u>almost</u> a Dirichlet integral $\int_0^{\delta} g(t)\,\frac{\sin(\alpha t)}{t}\,dt$,

but 't' in the denominator is replaced by $\sin\frac{t}{2}$. We can fix this by applying the Riemann-Lebesgue Lemma:

Note that

$$F(t) = \begin{cases} \dfrac{1}{t} - \dfrac{1}{2\sin t/2} & \text{if } 0 < t \le \pi \\[2ex] 0 & \text{if } t = 0 \end{cases}$$

is continuous on $[0,\pi]$. Therefore the RL-lemma gives

$$\lim_{n\to\infty}\int_0^{\pi}\left(\frac{1}{t} - \frac{1}{2\sin t/2}\right)\frac{f(x+t)+f(x-t)}{2}\,\sin((n+\tfrac{1}{2})t)\,dt = 0$$

and so

$$\lim_{n\to\infty}\int_0^{\pi}\frac{f(x+t)+f(x-t)}{2}\,\sin(n+\tfrac{1}{2})t\,dt$$

$$= \lim_{n\to\infty}\int_0^{\pi}\frac{f(x+t)+f(x-t)}{2}\,\frac{\sin(n+\frac{1}{2})t}{t}\,dt$$

Thus convergence of the Fourier series of f amounts to finding conditions on f that guarantee convergence of

$$\lim_{n\to\infty} \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} \, \frac{\sin(n+\frac{1}{2})t}{t}\, dt ,$$

and the Riemann-Lebesgue lemma lets us replace $\int_0^{\pi}$ with $\int_0^{\delta}$, because the $\int_{\delta}^{\pi}$ part tends to 0 as $n\to\infty$.

Thus we have:

<u>Theorem</u> (Riemann Localization theorem).

Assume $f\in L([0,2\pi])$ and suppose f has period 2π. Then the Fourier series of f converges at a point x_0 iff $\exists\ \delta>0$ $(\delta<\pi)$ such that

$$\lim_{n\to\infty} \int_0^{\delta} \frac{f(x+t)+f(x-t)}{2} \, \frac{\sin((n+\frac{1}{2})t)}{t}\, dt$$

exists, in which case the value of this limit is the value of the Fourier series.

<u>Remark</u>: This theorem is a bit surprising since it says that the convergence of

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

at $x=x_0$ depends only on the behaviour of f in a δ-nbhd of x_0. Yet the coefficients a_k, b_k depend on the behaviour of f over the interval $[0,2\pi]$!

Evidently each of the methods for proving convergence of the Dirichlet integrals (one by Jordan, one by Dini) will now apply to the limit in the previous theorem. The resulting theorems are Jordan's Test and Dini's Test

Jordan's Test:

Suppose $f \in L([0, 2\pi])$ is 2π-periodic. Fix $x \in [0, 2\pi]$ and define

$$g(t) = \frac{f(x+t) + f(x-t)}{2} \quad \text{for } t \in [0, \delta];$$

and let

$$S(x) = g(0^+) \text{ when it exists. Then.}$$

Thm: If f is BV on $[x-\delta, x+\delta]$ for some $\delta < \pi$, then $g(0^+) = S(x)$ exists, and the Fourier series generated by f converges at x to $g(0^+)$

Also

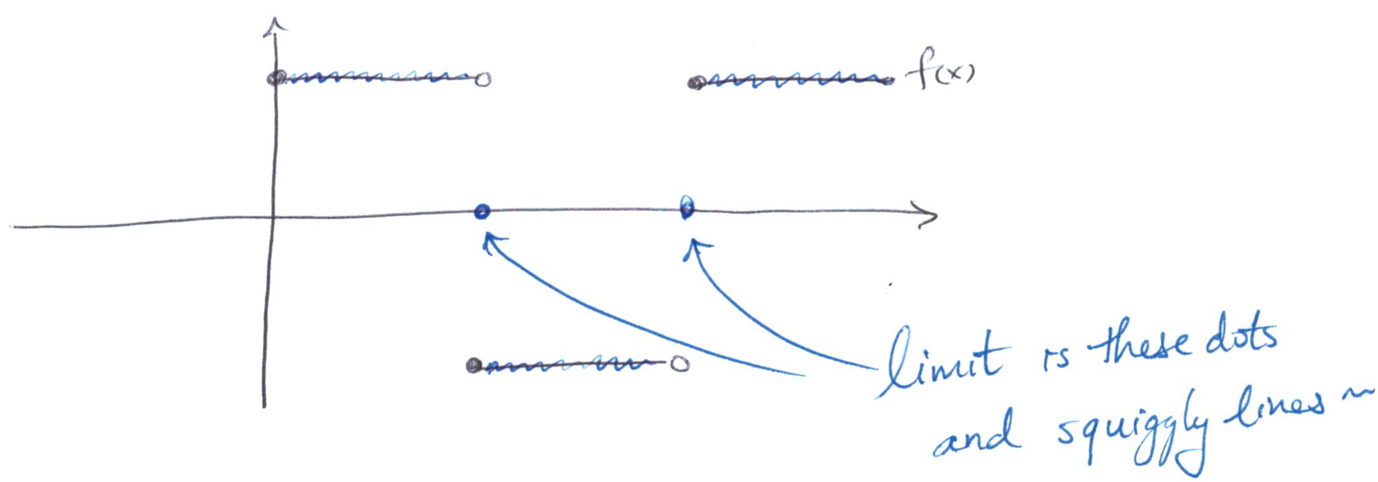
Dini's Test: With notation as above, if $g(0^+)$ exists and if $\int_0^\delta \frac{g(t) - S(x)}{t}\, dt$ exists for some $\delta < \pi$, Then the Fourier series generated by f converges at x to $S(x)$.

Example: Set $f(t) = \begin{cases} 1 & t \in [0, \pi) \\ -1 & t \in [\pi, 2\pi) \end{cases}$ extend to $\mathbb{R}$ in such a way as to be 2π-periodic.

Then f is of bounded variation on any compact interval, certainly on any $[x-\delta, x+\delta]$, $x \in \mathbb{R}$ and $\delta \in (0, \pi)$. Thus Jordan's test applies. Thus

$$\lim_{n\to\infty} S_n(x) = S(x) = \lim_{t \to 0^+} \frac{f(x+t) + f(x-t)}{2} = \begin{cases} f(x) & \text{if } x \neq k\pi,\ k \in \mathbb{Z} \\ 0 & \text{if } x = k\pi,\ k \in \mathbb{Z} \end{cases}$$

So



Remark: Even if $f(x)$ is continuous and 2π-periodic it may still have a Fourier series that diverges at a dense set of points. Examples are very difficult to construct, see e.g. Riesz products:

$$\prod_{k=1}^{\infty} \left(1 + i \frac{\cos 10^k x}{k}\right)$$

and a further result of Komolgorov:

"Une série de Fourier-Lebesgue divergente partout"
Comptes Rendus.

Therefore continuity does not imply convergence of a Fourier series, but it does imply Cesàro summability of the Fourier series. So we review Cesàro sums:

Definition: Let $s_n = \sum_{k=1}^{n} a_k$ be the partial sums of $\sum_{k=1}^{\infty} a_k$.

Define $\sigma_n = \dfrac{s_1 + s_2 + \dots + s_n}{n} = \dfrac{na_1 + (n-1)a_2 + \dots + 2a_{n-1} + a_n}{n}$

Then $\sum_{k=1}^{\infty} a_k$ is Cesàro summable or $(C,1)$-summable if $\{\sigma_n\}$ converges. In this case, call $s = \lim_{n\to\infty} \sigma_n$ the Cesàro sum of $\sum_{k=1}^{\infty} a_k = s \ (C,1)$.

Theorem: If $\sum_{k=1}^{\infty} a_k = s$, then $\sum_{k=1}^{\infty} a_k = s \ (C,1)$.

Proof: Page 206 in the text.

Example: The series $\sum_{k=1}^{\infty} (-1)^k$ diverges, since the partial sums are:

$$s_1 = -1$$
$$s_2 = -1+1 = 0$$
$$s_3 = -1+1-1 = -1$$
$$s_4 = 0 \ \dots \text{etc.}$$

But the σ_n's are:

$$\sigma_1 = \frac{-1}{1} = -1 \qquad \sigma_3 = \frac{-1+0-1}{3} = -\frac{2}{3}$$

$$\sigma_2 = \frac{-1+0}{2} = -\frac{1}{2} \qquad \sigma_4 = \frac{-1+0-1+0}{4} = -\frac{1}{2}$$

and in general

$$\sigma_{2n+1} = \frac{-(n+1)}{2n+1}$$

$$\sigma_{2n} = \frac{-1}{2},$$

so $\lim_{n\to\infty} \sigma_n = \frac{-1}{2}$, and $\sum_{k=1}^{\infty} (-1)^k = -\frac{1}{2} \quad (C,1)$.

<u>Theorem 11.14</u> Assume that $f \in L([0,2\pi])$ and suppose that f is periodic with period 2π. Let s_n denote the usual partial sums and

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + \dots + s_{n-1}(x)}{n} \quad (n = 0, 1, 2, \dots).$$

Then

$$\sigma_n(x) = \frac{1}{n\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)} \, dt$$

<u>Proof</u>: Use the integral representation of $s_n(x)$:

$$s_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} D_n(t) \, dt$$

and substitute it into the defining formula of $\sigma_n(x)$.

Using the formula

$$\sum_{k=1}^{n} \sin(2k-1)x = \frac{\sin^2 nx}{\sin x}$$

allows us to convert the result into the claimed formula.

<u>Remarks</u>: ① The quantity $\frac{\sin^2(n\frac{1}{2}t)}{\sin(\frac{1}{2}t)}$ is called Fejer's kernel

(2) If we take $f(x) = 1$, the constant function, then $\sigma_n = S_n = 1$ for each n (compute the Fourier coeffs and check they're all 0 (except the first)) and we arrive at

$$\frac{1}{n\pi}\int_0^\pi \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt = 1.$$

Thus, for any $s \in \mathbb{R}$ we can write

$$\sigma_n(x) - s = \frac{1}{n\pi}\int_0^\pi \left(\frac{f(x+t)+f(x-t)}{2} - s\right)\frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt \quad (*)$$

therefore if we succeed in finding s such that

$$\lim_{n\to\infty}\frac{1}{n\pi}\int_0^\pi \left(\frac{f(x+t)+f(x-t)}{2} - s\right)\frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt = 0$$

then it follows that $\lim_{n\to\infty}\sigma_n = s$, and we'll have found the Cesàro sum! The next theorem tells us how to choose s (in a fashion depending on x).

<u>Theorem</u>: (Fejèr) Assume $f \in L([0,2\pi])$ and suppose that f is periodic with period 2π. Define

$$s(x) = \lim_{t\to 0^+}\frac{f(x+t)+f(x-t)}{2}$$

whenever the limit exists. Then whenever $s(x)$ is defined the Fourier series of f is $(C,1)$ summable and

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx) = s(x) \quad ((C,1))$$