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§11.5 Continued.

We just saw that if $f(x) \sim \sum_{k=0}^{\infty} c_k \varphi_k$ then

$$\sum_{n=0}^{\infty} |c_n|^2 \leq \|f\|^2 \quad \text{(Bessel's formula)}$$

and

$$\sum_{n=0}^{\infty} |c_n|^2 = \|f\|^2 \quad \text{(Parseval's formula)}$$

iff

$$\lim_{n\to\infty} \left\| f - \sum_{k=0}^{n} c_k \varphi_k \right\| = 0.$$

Consequences of these inequalities/equalities; observations

① Since $\sum_{n=0}^{\infty} |c_n|^2$ converges, $|c_n|^2 \to 0$ as $n \to \infty$, and thus $c_n \to 0$ as $n \to \infty$ as well. Applying this observation to the orthonormal system $\varphi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$, we get

$$0 = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \int_0^{2\pi} f(x) \frac{e^{-inx}}{\sqrt{2\pi}}\, dx$$

$$\Rightarrow \lim_{n\to\infty} \int_0^{2\pi} f(x) e^{-inx}\, dx = 0.$$

Using Euler's formula:

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\left[\cos(nx) + i\sin(nx)\right] dx = 0$$

which implies the real and imaginary parts must converge to zero independantly:

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\cos(nx)\,dx = \lim_{n\to\infty} \int_0^{2\pi} f(x)\sin(nx)\,dx = 0.$$

(2) Parseval's formula is a "generalization" of the Pythagorean Theorem, instead of, for $\bar{x} \in \mathbb{R}^n$

$$\|\vec{x}\|^2 = x_1^2 + x_2^2 + \dots + x_n^2 \quad \text{we have}$$

$$\|f\|^2 = |c_1|^2 + |c_2|^2 + \dots \quad \text{for } f \in L^2(I).$$

(3) We already saw that the collection

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(3x)}{\sqrt{\pi}}, \dots \right\} \text{ is orthonormal}$$

on $[0, 2\pi]$. So we would compute the fourier coefficients for these functions using the formulae

$$c_{odd} = \left(f, \frac{\cos(nx)}{\sqrt{\pi}}\right) = \int_0^{2\pi} f(x) \frac{\cos(nx)}{\sqrt{\pi}}\, dx \quad \text{and}$$

$$c_{even} = \left(f, \frac{\sin(nx)}{\sqrt{\pi}}\right) = \int_0^{2\pi} f(x) \frac{\sin(nx)}{\sqrt{\pi}}\, dx, \text{ as expected.}$$

If instead of $f \sim \sum c_k \varphi_k$ for φ_k as above we decided to write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

then the a_n's and b_n's are no longer the Fourier coefficients as defined in our text, as the functions $\cos nx$, $\sin nx$ are not orthonormal. Here, a_n is related to the n^{th} Fourier coefficient (n odd), by $a_n = \frac{c_n}{\sqrt{\pi}}$ (similarly $b_n = \frac{c_n}{\sqrt{\pi}}$) and so are computed as

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx.$$

Then Parseval's formula would become

$$\cancel{\pi}\left(\sum_i |a_k|^2 + \sum_i |b_k|^2 \right) = \|f\|^2.$$

In general, working on an interval of length $[0, 2L]$ for some $L > 0$, the functions become

$$\left\{ \frac{1}{\sqrt{2L}}, \frac{\cos\left(\frac{n\pi x}{L}\right)}{\sqrt{L}}, \frac{\sin\left(\frac{n\pi x}{L}\right)}{\sqrt{L}} \right\}$$ giving us a new

orthonormal system on $[0, 2L]$. Then the a_n's and b_n's are computed by

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

and $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$. Then these a_n's and b_n's are related to the "true" Fourier coefficients c_k, as before, by $\frac{c_k}{\sqrt{L}} = a_k$, $\frac{c_k}{\sqrt{L}} = b_k$ for k odd/even respectively. Parseval's formula is $\sum |a_k|^2 + \sum_i |b_k|^2 = \|f\|^2 \cdot \frac{1}{L}$.

In fact, there is nothing special about the interval $[0, 2L]$ — we can use any interval $[0+y, 2L+y]$, $y \in \mathbb{R}$ (any shift). Then, for example (provided we check the necessary convergence properties) Parseval's formula gives remarkable results.

Consider $f(x) = x$ on the interval $[-2, 2]$. Compute a_n and b_n using the formulas above. We find $a_n = 0$ since $f(x)$ is odd, and

$$b_n = \frac{1}{2}\int_{-2}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx = \text{some work} \dots$$

$$= \frac{-4}{n\pi}\cos(n\pi), = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ \frac{-4}{n\pi} & n \text{ even} \end{cases} = \frac{(-1)^{n+1} \cdot 4}{n\pi}.$$

Then Parseval's formula becomes

$$\cancel{\sum_1 |a_k|^2} + \sum_1 |b_k|^2 = \|f\|^2 \cdot \frac{1}{2}$$

$$\Rightarrow \left(\frac{4}{\pi}\right)^2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \int_{-2}^{2} x^2 dx \cdot \frac{1}{2}$$

And $\displaystyle\int_{-2}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{16}{3}$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{\cancel{16}} \cdot \frac{\cancel{16}}{3} \cdot \frac{1}{2} = \frac{\pi^2}{6}.}$$

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§ 11.6 The Riesz-Fischer Theorem

First we must pick up some background material that was not covered in Analysis 2. Unfortunately we cannot provide a proof as it would represent a substantial digression.

Theorem 10.57: Let $\{f_n\}$ be a Cauchy sequence of complex-valued functions with $f_n \in L^2(I)$ for all n. That is, $\forall \varepsilon > 0 \; \exists N$ s.t.

$$\| f_m - f_n \| < \varepsilon \quad \text{whenever } m, n \geq N$$

(here $\|\cdot\|$ is the norm on $L^2(I)$). Then there exists a function $f \in L^2(I)$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$.

Interpretation of this theorem: Recall that a metric space (X, d) is complete if every Cauchy sequence converges. The set $L^2(I)$ with

$$d(f,g) = \|f - g\| = \cancel{\int_I f(x)\overline{g(x)}\,dx} \left(\int_I |f - g|^2 dx \right)^{1/2}$$

is very nearly a metric space. The function d satisfies:

1. $d(f,f) = 0$
2. $d(f,g) = d(g,f)$
3. $d(f,h) \leq d(f,g) + d(g,h)$

BUT NOT

4. $d(f,g) > 0$ if $f \neq g$, because f and g can differ on a set of measure zero and still yield $f - g = 0$ almost everywhere, so $\int_I |f-g|^2 dx = 0$.

Thus we call $L^2(I)$ a *semimetric space*. The Riesz Fischer Theorem tells us that $L^2(I)$ is *complete*, in the same sense as a metric space (X, d).

We use this to prove:

Theorem: Assume $\{\varphi_0, \varphi_1, \ldots\}$ are orthonormal on I. Let $\{c_n\}$ be any sequence such that $\sum_{n=0}^{\infty} |c_n|^2 < \infty$. Then $\exists f \in L^2(I)$ such that

a) $(f, \varphi_k) = c_k \quad \forall k \geq 0$, and

b) $\|f\|^2 = \sum_{k=0}^{\infty} |c_k|^2$

Remark: This fact genuinely relies on $L^2(I)$ being complete. There is no analagous theorem for functions f st f^2 is Riemann-integrable, for instance (The RF theorem does not apply to such functions).

Proof: Set $s_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$. We will show that $\exists f \in L^2(I)$ with $(f, \varphi_k) = c_k$ and $\lim_{n\to\infty} \|f - s_n\| = 0$. Part (b) of the Theorem then follows from Parseval's formula.

First, if $m > n$ then $s_n - s_m = \sum_{k=n+1}^{m} c_k \varphi_k$ and so

$$\|s_n - s_m\| = \sum_{k=n+1}^{m} \sum_{r=n+1}^{m} c_k \bar{c}_r (\varphi_k, \varphi_r)$$

$$= \sum_{k=n+1}^{m} |c_k|^2.$$

Since $\lim_{k\to\infty} |c_k|^2 = 0$, this sum can be made arbitrarily small by choosing n and m sufficiently large. Thus $\{s_n\}$ is a Cauchy sequence in $L^2(I)$. By the Riesz Fischer Theorem, there exists $f \in L^2(I)$ such that

$$\lim_{n\to\infty} \|f - s_n\| = 0.$$

Thus we need only show that $(f, \varphi_k) = c_k$ for such an f. Note that for $n \geq k$, $(s_n, \varphi_k) = c_k$ and so by the Cauchy-Schwartz inequality:

$$|c_k - (f, \varphi_k)| = |(s_n, \varphi_k) - (f, \varphi_k)|$$

$$= |(s_n - f, \varphi_k)| \leq \|s_n - f\| \cdot \underbrace{\|\varphi_k\|}_{= 1}$$

and $\lim_{n\to\infty} \|s_n - f\| = 0$ gives the result.

§11.7

We now specialize and consider the trig orthonormal system, and the Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}{}' (a_n \cos(nx) + b_n \sin(nx)) \quad \text{on } [0, 2\pi].$$

We focus on:

① Convergence problem: Does the series on the RHS above converge, at least at some point $x_0 \in I$?

② Representation problem: When the anwer to ① is 'yes', do we have $f(x_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx_0) + b_n \sin(nx_0))$?

In order to address either of these questions — which we will do by proving Fejér's theorems — we first need some preliminary limit results.

The Riemann–Lebesgue Lemma (§11.8)

Theorem: Assume $f \in L(I)$. Then, for all $\beta \in \mathbb{R}$ we have

$$\lim_{\alpha \to \infty} \int_I f(t) \sin(\alpha t + \beta)\, dt = 0.$$

Proof: Recall the characteristic function of a set $S \subseteq \mathbb{R}$ is

$$\chi_S : \mathbb{R} \to \{0, 1\}$$

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

If $f(t)$ happens to be the characteristic function of an interval $[a,b]$, then $([a,b] \subseteq I)$

$$\int_I f(t)\sin(\alpha t+\beta)\,dt = \int_a^b \sin(\alpha t+\beta)\,dt$$

and thus

$$\left|\int_a^b \sin(\alpha t+\beta)\,dt\right| \doteq \left|\frac{\cos(a\alpha+\beta)-\cos(b\alpha+\beta)}{\alpha}\right| \leq \frac{2}{\alpha} \quad (\text{if } \alpha>0)$$

and thus $\lim\limits_{\alpha\to 0} \int_I f(t)\sin(\alpha t+\beta)\,dt = 0$.

The result is also true if $f(t)$ is constant on (a,b), zero on $\mathbb{R}\setminus[a,b]$ and arbitrary at a and b. In particular the result holds for every step function.

But now the result holds for every $f \in L(I)$:
Let $\varepsilon>0$ be given, then there exists a step function s such that

$$\int_I |f-s| < \frac{\varepsilon}{2} \quad (\text{Theorem } 10.19)$$

(Recall a step function on $[a,b]$ is a function for which $\exists\ \{x_0, x_1, \ldots, x_n\} \subset [a,b]$ s.t. the function is constant on each interval (x_i, x_{i+1})).

Then $\exists\ M>0$ s.t.

$$\left|\int_I s(t)\sin(\alpha t+\beta)\,dt\right| < \frac{\varepsilon}{2} \text{ if } \alpha \geq M,$$

since our result holds for step functions.

So if $\alpha \geq M$,

$$\left|\int_I f(t)\sin(\alpha t+\beta)\,dt\right| \leq \left|\int_I (f(t)-s(t))\sin(\alpha t+\beta)\,dt\right| + \left|\int_I s(t)\sin(\alpha t+\beta)\,dt\right|$$

$$\leq \int_I |f(t)-s(t)|\,dt + \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Remark ① The same result obviously holds if sin is replaced with cosine, by taking $\beta' + \pi/2 = \beta$; so

$$\sin(\alpha t+\beta'+\pi/2) = \cos(\alpha t+\beta').$$

② By taking either $\beta=0$ or $\beta=\pi/2$ we also arrive at

$$\lim_{\alpha\to\infty}\int_I f(t)\sin(\alpha t)\,dt = \lim_{\alpha\to\infty}\int_I f(t)\cos(\alpha t)\,dt = 0.$$

③ The claim in the previous proof that "the result holds for all step functions" easily follows from:

If f is a step function then $f = \sum_{k=1}^{n} c_k f_k(t)$ where f_k are characteristic functions of intervals. Then

$$\lim_{\alpha\to\infty}\int_I \left(\sum_{k=1}^{n} c_k f_k(t)\right)\sin(\alpha t+\beta)\,dt$$

$$= \sum_{k=1}^{n} c_k \lim_{\alpha\to\infty}\int_I f_k(t)\sin(\alpha t+\beta)\,dt = 0.$$

Corollary (Needed for later). (Theore 11.7).

If $f \in L(-\infty, \infty)$ then

$$\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} f(t) \frac{1-\cos(\alpha t)}{t} \, dt = \int_0^\infty \frac{f(t)-f(-t)}{t} \, dt.$$

Proof: First note that $\frac{1-\cos(\alpha t)}{t}$ is continuous and bounded on $(-\infty, \infty)$ if we understand this expression to take on $\lim_{t \to 0} \frac{1-\cos(\alpha t)}{t} = 0$ at $t=0$, and thus the Lebesgue integral on the LHS exists. Then compute:

$$\int_{-\infty}^{\infty} f(t) \frac{1-\cos(\alpha t)}{t} \, dt = \int_0^\infty f(t) \frac{1-\cos(\alpha t)}{t} dt + \int_{-\infty}^{0} f(t) \frac{1-\cos \alpha t}{t} dt$$

$$= \int_0^\infty (f(t)-f(-t)) \frac{1-\cos(\alpha t)}{t} dt$$

$$= \int_0^\infty \frac{f(t)-f(-t)}{t} dt - \int_0^\infty \frac{f(t)-f(-t)}{t} \cos(\alpha t) dt$$

Thus, taking limits as $\alpha \to \infty$, (and employing Remark 2) we arrive at the claimed result.

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§ 11.9 Dirichlet Integrals.

Next, we prep a few integrals that will be useful going forward. The integral

$$\int_0^\delta g(t)\,\frac{\sin \alpha t}{t}\,dt\ , \quad \alpha \in \mathbb{R}$$

is called a Dirichlet integral. Suppose that $\lim_{t\to 0^+} g(t) = g(0^+)$ exists, in fact, suppose for the time being that $g(t)$ has the constant value $g(0^+)$ on $[0,\delta]$.

Then
$$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt$$

$$= g(0^+)\left[\int_0^\infty \frac{\sin(\alpha t)}{t}\,dt\right]\frac{2}{\pi} = g(0^+)\cdot\frac{\pi}{2}\cdot\frac{2}{\pi} = g(0^+).$$

See Example 3, §10.16

More generally, suppose $g \in L([0,\delta])$ and $0<\varepsilon<\delta$.

Then
$$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_\varepsilon^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt = 0,$$

by the Riemann-Lebesgue Lemma. So the value of the limit

$$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt\,,$$

which we will now investigate, seems to

be dictated by the behaviour of $g(t)$ near 0 — and should be equal to $g(0^+)$.

<u>Theorem</u> 11.8 (Jordan). If g is of bounded variation on $[0,\delta]$, then $\displaystyle\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt = g(0^+)$.

(Recall a function is of bounded variation if

$$V(f) = \sup_{P\in\mathcal{P}(I)} \sum_{k=0}^{n_p-1} |f(x_{i+1}) - f(x_i)| \quad \text{exists, where } \mathcal{P}(I)$$

is the set of all partitions $\{x_0,\dots,x_{n_p}\}$ of I).

<u>Proof</u>: It suffices to prove the theorem for increasing function $g(t)$, since by Theorem 6.13 every function of bounded variation is a difference of increasing functions.

So suppose $g(t)$ is increasing on $[0,\delta]$, $\alpha>0$ and $0<h<\delta$. Then

$$\int_0^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt = \int_0^h \big(g(t)-g(0^+)\big)\frac{\sin(\alpha t)}{t}\,dt$$

$$+ \int_0^h g(0^+)\,\frac{\sin(\alpha t)}{t}\,dt + \int_h^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt.$$

Call the quantities on the RHS $I_1(\alpha,h)$, $I_2(\alpha,h)$ and $I_3(\alpha,h)$ respectively. Now the RL-Lemma applies to $I_3(\alpha,h)$ and

$$\lim_{\alpha\to\infty} I_3(\alpha,h) = 0.$$

We compute directly

$$I_2(\alpha,h) = g(0^+)\int_0^h \frac{\sin(\alpha t)}{t}\,dt$$

$$= g(0^+)\int_0^{h\alpha} \frac{\sin t}{t}\,dt$$

To check this, set $u=\alpha t \Rightarrow t=\frac{u}{\alpha}$ & $dt=\frac{du}{\alpha}$

So $$\lim_{\alpha\to\infty} I_2(\alpha,h) = \lim_{\alpha\to\infty} g(0^+)\int_0^{h\alpha}\frac{\sin t}{t}\,dt = \frac{\pi}{2}g(0^+).$$

Thus we need only deal with $I_1(\alpha,h)$.

Choose M with $\left|\int_a^b \frac{\sin t}{t}\,dt\right| < M \quad \forall a,b$ with $0 \le a \le b$.

Then $\left|\int_a^b \frac{\sin(\alpha t)}{t}\,dt\right| < M \quad \forall \alpha > 0$. Now let $\varepsilon > 0$ and

choose $h \in (0,\delta)$ s.t. $|g(h)-g(0^+)| < \frac{\varepsilon}{3M}$. Then

$g(t)-g(0^+) \geq 0$ if $t\in[0,h]$.

Bonnet's Theorem (7.37) lets us write

$$I_1(\alpha,h) = \int_0^h (g(t)-g(0^+))\frac{\sin(\alpha t)}{t}\,dt = (g(h)-g(0^+))\int_c^h \frac{\sin(\alpha t)}{t}\,dt$$

for some $c\in[0,h]$. By our choice of h, we then get

$$|I_1(\alpha,h)| = |g(h)-g(0^+)|\left|\int_c^h \frac{\sin(\alpha t)}{t}\,dt\right| < \frac{\varepsilon}{3M}\cdot M = \frac{\varepsilon}{3}.$$

For the same h, choose A s.t. $\alpha \geq A$ implies

$$|I_3(\alpha,h)| < \frac{\varepsilon}{3} \text{ and } \left|I_2(\alpha,h) - \frac{\pi}{2}g(0^+)\right| < \frac{\varepsilon}{3}.$$

Then with h as above, for $\alpha \geq A$ we have

$$\left| \int_0^\delta g(t) \frac{\sin(\alpha t)}{t} dt - \frac{\pi}{2} g(0^+) \right| < \varepsilon,$$

which proves the Theorem.

We also have a simpler:

Theorem (Dini): Assume $g(0^+)$ exists and that $\exists\, \delta > 0$ s.t. the Lebesgue integral

$$\int_0^\delta \frac{g(t) - g(0^+)}{t} dt \text{ exists.}$$

Then $$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin(\alpha t)}{t} dt = g(0^+).$$

Proof:

$$\int_0^\delta g(t) \frac{\sin(\alpha t)}{t} dt = \int_0^\delta \frac{g(t) - g(0^+)}{t} \sin(\alpha t) dt + g(0^+) \int_0^\delta \frac{\sin t}{t} dt.$$

Then as $\alpha \to \infty$; the first term $\to 0$ by RL Lemma, and the second $\to \frac{\pi}{2} g(0^+)$,

so the Theorem follows.

<u>Remarks</u>: • Neither condition implies the other

• Dini's condition is implied by: If $\exists$ positive M, p such that

$$|g(t) - g(0^+)| < M t^p \quad \forall t \in (0, \delta]$$

in particular, this holds with $p = 1$ whenever g has a right hand derivative at 0, ie

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$