MATH 3472

Examples (applications of convolution):

Recall (or maybe you're seeing it for the first time) that the gamma function is an extension of factorials to all of $\mathbb{C}$. That is,

$$\Gamma(n) = (n-1)!$$

yet $\Gamma(z)$ is defined $\forall z \in \mathbb{C}$.

It is given by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx$$

and comes up often in probability/statistics.

It is sometimes called an Euler integral of the second kind. An Euler integral of the first kind is: the beta function:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

and is related to the gamma function as follows:

Set
$$f_p(t) = \begin{cases} t^{p-1}e^{-t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Then $f \in L(\mathbb{R})$ and

$$\int_{-\infty}^\infty f_p(t)\,dt = \int_0^\infty t^{p-1}e^{-t}\,dt = \Gamma(p)$$

If we set $h = f_p * f_q$ then applying

$$\int_{-\infty}^{\infty} h(x) e^{-ixu}\, dx = \left(\int_{-\infty}^{\infty} f_p(t) e^{-itu} dt\right)\left(\int_{-\infty}^{\infty} f_q(t) e^{-itu} dt\right)$$

with $u=0$ gives

$$\int_{-\infty}^{\infty} h(x) = \Gamma(p)\Gamma(q).$$ (This requires $p \geq 0$, $q \geq 0$ so that $f_p, f_q \in L(\mathbb{R})$)

On the other hand, we can also calculate the LHS more directly: If $x > 0$ then

$$h(x) = \int_0^x f_p(t) f_q(x-t)\, dt = \int_0^x t^{p-1} e^{-t} (x-t)^{q-1} e^{-(x-t)} dt$$

$$= e^{-x} \int_0^x t^{p-1}(x-t)^{q-1} dt$$

while $h(x) = 0$ if $x < 0$. Set $t = ux$, then for $x > 0$
(both vanish on neg axis)

$$h(x) = e^{-x} \int_0^1 (ux)^{p-1}(x-ux)^{q-1}\, x\, du$$

(used $dt = x\, du$)

$$= e^{-x} \cdot x^{p-1} \cdot x^{q-1} \cdot x \int_0^1 u^{p-1}(1-u)^{q-1} du$$

$$= e^{-x} x^{p+q-1} B(p,q).$$

So now

$$\int_{-\infty}^{\infty} h(x) \cdot dx = \int_{-\infty}^{\infty} e^{-x} x^{p+q-1} B(p,q)\, dx = B(p,q)\Gamma(p+q)$$

Overall, by equating our two expressions for $\int_{-\infty}^{\infty} h(x)dx$ we have

$$\Gamma(p)\Gamma(q) = B(p,q)\Gamma(p+q)$$

$$\Rightarrow B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

For all $p \geq 0$, $q \geq 0$

Poisson summation:

Poisson summation provides a relationship between the Fourier series of a function and its Fourier transform.

Theorem 11.24: Let f be a nonnegative function s.t. $\int_{-\infty}^{\infty} f(x)\,dx$ exists as an improper Riemann integral. Assume f is increasing on $(-\infty, 0]$ and decreasing on $[0,\infty)$. Then

$$\sum_{m=-\infty}^{\infty} \frac{f(m+)+f(m-)}{2} = \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(t)e^{-2\pi i n t}\,dt}_{\text{exp. Fourier transform at } x=n\cdot 2\pi}$$

(Proof next class).

Another formulation, more common in other literature:

Theorem: Let f be a continuous function for which $|f(x)|$, $|f'(x)|$ and $|f''(x)|$ exist for all $x \in \mathbb{R}$ and are all bounded by $c(1+|x|)^{-2}$ for some fixed c. Then writing $\hat{f}(x)$ for the Fourier transform $\mathcal{F}(f)$, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Remark: This assumes a definition of

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\, dx$$

Without this normalization it gives

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(2n\pi)$$ (a special case of our formula)

Application:

Consider $f(x) = \dfrac{1}{x^2+a^2}$. We would like to calculate its Fourier transform, $\mathcal{F}(f(x))$. Here is a trick:

Let $b(x) = e^{-a|x|}$. Then if $a>0$.

$$\begin{aligned}
\mathcal{F}(b(x)) &= \int_{-\infty}^{\infty} e^{-a|x|} e^{-ixy}\, dx \\
&= \int_{-\infty}^{0} e^{ax} e^{-ixy}\, dx + \int_{0}^{\infty} e^{-ax} e^{-ixy}\, dx \\
&= \left[\frac{e^{(a-iy)x}}{a-iy}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+iy)x}}{a+iy}\right]_{0}^{\infty} \\
&= \frac{1}{a-iy} + \frac{1}{a+iy} = \frac{2a}{a^2+y^2}
\end{aligned}$$

Then the Fourier inversion formula gives

$$e^{-|x|a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+y^2} e^{iyx}\, dy$$

Replacing x by $-x$ and multiplying by $\frac{\pi}{a}$ gives

$$\frac{\pi}{a} e^{-a|x|} = \int_{-\infty}^{\infty} \frac{e^{-iyx}}{a^2+y^2}\, dy = \mathcal{F}(f).$$

So the Poisson formula gives:

$$\sum_{n\in\mathbb{Z}} f(n) = \left(\sum_{n=1}^{\infty} \frac{2}{n^2+a^2}\right) + \frac{1}{a^2}$$

$$= \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n) = \left(\sum_{n=1}^{\infty} \frac{2\pi}{a} e^{-a(2\pi n)}\right) + \frac{\pi}{a}$$

$$= \frac{2\pi}{a}\sum_{n=1}^{\infty} r^n + \frac{\pi}{a} \quad (r = e^{-2\pi a})$$

$$= \frac{2\pi}{a}\left(\frac{1}{1-e^{-2\pi a}}\right) + \frac{\pi}{a} - \frac{2\pi}{a}$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\left(\frac{1}{1-e^{-2\pi a}}\right) - \frac{\pi}{2a} - \frac{1}{2a^2}$

So, e.g $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{1-e^{-2\pi}} - \frac{\pi}{2} - \frac{1}{2}$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{1-e^{-2\pi}} + \frac{-\pi+1}{2}$$

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Recall we were about to prove:

<u>Theorem</u>: Let f be a nonnegative function such that $\int_{-\infty}^{\infty} f(x)\,dx$ exists as an improper Riemann integral. Assume f increases on $(-\infty, 0]$ and decreases on $[0,\infty)$. Then
$$\sum_{n\in\mathbb{Z}}' \frac{f(n+)+f(n-)}{2} = \sum_{n\in\mathbb{Z}}' \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i n t}\,dt ,$$
each being absolutely convergent.

(Note: RHS is $\mathcal{F}(f)(2\pi n)$, summed over $n\in\mathbb{Z}$).

<u>Proof</u>: Define a function F by
$$F(x) = \sum_{m\in\mathbb{Z}}' f(m+x).$$

<u>Claim</u>: The series above converges absolutely $\forall x\in\mathbb{R}$, and convergence is uniform on $[0,1]$.

<u>Proof of claim</u>: As $f(x)$ is decreasing on $[0,\infty)$, we have

<u>if $x\ge 0$</u>: $\displaystyle\sum_{m=0}^{N}' f(m+x) \le f(0) + \sum_{m=1}^{N}' f(m) \le f(0) + \int_0^{\infty} f(t)\,dt$

rectangles of width 1 under $f(x)$ from $x=0$ to $x=N$.

Since $\int_0^{\infty} f(t)\,dt$ exists by assumption, the Weierstrass M-test gives uniform convergence of $\sum_{m=0}^{\infty}' f(m+x)$ on $[0,\infty)$.

Similarly we may show that $\sum_{m=-\infty}^{-1} f(m+x)$ converges uniformly on $(-\infty, 1]$: If $x<1$ then

$$\sum_{m=-\infty}^{-1} f(m+x) \leq \sum_{m=-\infty}^{-1} f(m+1)$$

$$= \sum_{m=-\infty}^{0} f(m) \leq f(0) + \int_{-\infty}^{0} f(x)\,dx.$$

need $x<1$ here

So again by the Weierstrass M-test, the series converges uniformly on $[-\infty, 1]$. Thus $\sum_{n\in\mathbb{Z}} f(n+x)$ converges for all x, and convergence is uniform on

$$(-\infty, 1] \cap [0, \infty) = [0, 1].$$

Thus the claim holds.

Next, note that the sum is periodic with period 1:

$$F(x+1) = \sum_{n\in\mathbb{Z}} f(n+(x+1)) = \sum_{n\in\mathbb{Z}} f(n+x) = F(x),$$ note

that this step uses the fact that the RHS is a rearrangement of the LHS, and the series converges absolutely.

Finally we'll see that F is BV on every compact interval. Trick:

If $0 \leq x \leq \frac{1}{2}$, then $f(m+x)$ is a decreasing function of x if $m>0$, and an increasing function of x if $m<0$.

So we can write:

$$F(x) = \sum_{m=0}^{\infty} f(m+x) - \sum_{m=-\infty}^{-1} (-f(m+x))$$

So $F(x)$ is a difference of two decreasing functions, so is BV on $[0, \frac{1}{2}]$

(Remark: This required $f_n(x)$ decreasing $\Rightarrow \sum_{n=0}^{\infty} f_n(x)$ decreasing, which is a nontrivial fact to check).

We can argue similarly + show $f(x)$ is BV on $[-\frac{1}{2}, 0]$, and then by periodicity $F(x)$ is BV on every compact interval.

Now consider the Fourier series of $F(x)$, in exponential form:

$$F(x) \sim \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$$

where $a_n = \int_0^1 F(x) e^{-2\pi i n x} dx$, this uses the fact that F is BV $\Rightarrow F$ is Riemann integrable on $[0, 1]$.

Now we may apply Jordan's test to conclude that the Fourier series converges for every x, and

$$\frac{F(x+) + F(x-)}{2} = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$$

Let ~~us~~ us express the coefficients a_n in another way to get the Poisson summation formula.

$$a_n = \int_0^1 F(x)\, e^{-2\pi i n x}\, dx$$

$$= \int_0^1 \left(\sum_{m \in \mathbb{Z}}' f(m+x) \right) e^{-2\pi i n x}\, dx$$

$$= \sum_{m \in \mathbb{Z}}' \int_0^1 f(m+x) e^{-2\pi i n x}\, dx \quad \text{(by uniform convergence)}$$

Set $t = m + x$

$$= \sum_{m \in \mathbb{Z}}' \int_m^{m+1} f(t)\, e^{-2\pi i n t}\, dt$$

note here: $e^{-2\pi i n m t} = 1$ made this expression simpler after substitution.

$$= \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t}\, dt.$$

So if we plug this in place of a_n in the eqn arising from Jordan's test:

$$\frac{F(x+) + F(x-)}{2} = \sum_{n \in \mathbb{Z}}' \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t}\, dt \cdot e^{2\pi i n x}$$

Set $x = 0$. Then

$$\frac{F(0+) + F(0-)}{2} = \sum_{n \in \mathbb{Z}}' \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i n t}\, dt$$

Note that

$$\frac{F(0+) + F(0-)}{2} = \frac{1}{2} \left(\sum_{m \in \mathbb{Z}}' f(m + 0^+) + \sum_{m \in \mathbb{Z}}' f(m + 0^-) \right)$$

$$= \sum_{m \in \mathbb{Z}}' \frac{f(m+) + f(m-)}{2}$$

and the theorem follows.

<u>Remarks</u>: • We already saw that when f is cts the left hand side becomes

$$\sum_{n\in\mathbb{Z}}' f(n).$$

• The monotonicity requirements on $f(x)$ can be relaxed. For example: The conclusion of the theorem:

$$\sum_{n\in\mathbb{Z}} \frac{f(n+)+f(n-)}{2} = \sum_{n\in\mathbb{Z}}' \int_{-\infty}^{\infty} f(t)e^{-2\pi i n t}\,dt$$

depends linearly on $f(x)$. So if $f(x) = a\,g(x) + b\,h(x)$, where $g(x)$, $h(x)$ satisfy the hypotheses of the theorem (but $f(x)$ does not), then we still get

$$\sum_{n\in\mathbb{Z}}' \frac{f(n+)+f(n-)}{2} = \sum_{n\in\mathbb{Z}}' \int_{-\infty}^{\infty} f(t)e^{-2\pi i n t}\,dt.$$

In particular, this means that if $f = u + iv$, where u and v are real-valued functions of real variables, then the theorem holds for f.

<u>Example</u>: The hyperbolic cotangent is

$$\coth(x) = \frac{e^{2x}+1}{e^{2x}-1}, \quad \text{and}$$

$$\coth(x) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}' \frac{1}{x^2+(\pi n)^2}.$$

Then from last day:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\left(\frac{1}{1-e^{-2\pi a}}\right) - \frac{\pi}{2a} - \frac{1}{2a^2}$$

Choose $a = \frac{x}{\pi}$. Then

$$\sum_{n=1}^{\infty} \frac{\pi^2}{\pi^2\left(n^2+\left(\frac{x}{\pi}\right)^2\right)} = \frac{\pi}{\frac{x}{\pi}}\left(\frac{1}{1-e^{-2\pi\left(\frac{x}{\pi}\right)}}\right) - \frac{\pi}{2\left(\frac{x}{\pi}\right)} - \frac{1}{2\left(\frac{x}{\pi}\right)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 + x^2} = \frac{1}{\pi^2}\left(\frac{\pi^2}{x}\left(\frac{1}{1-e^{-2x}}\right) - \frac{\pi^2}{2x} - \frac{\pi^2}{2x^2}\right)$$

$$\Rightarrow 2x\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 + x^2} = \frac{2}{1-e^{-2x}} - 1 - \frac{1}{x}$$

$$\Rightarrow \frac{1}{x} + 2x\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2+x^2} = \frac{2-(1-e^{-2x})}{1-e^{-2x}} = \frac{1+e^{-2x}}{1-e^{-2x}}.$$

$$= \coth(x).$$

(sign mistake?)

nope

$$\frac{1+e^{-2x}}{1-e^{-2x}} = \frac{e^{2x}+1}{e^{2x}-1} \quad \text{(check).}$$

MATH 342

Final lecture: A summary of Fourier series.

Fix a collection $\{\varphi_0, \varphi_1, \dots\}$, an orthonormal system on an interval I. The Fourier coefficients

$$c_k = (f, \varphi_k)$$

of a function f are so defined because the partial sums

$$S_n(x) = \sum_{k=0}^{n} c_k \varphi_k$$

best approximate f in the mean. I.e if $\sum_{k=0}^{n} b_k \varphi_k = t_n$ is any other fun, then

$$\| f - s_n \| \leq \| f - t_n \|$$

with equality iff $c_k = b_k$.

For general Fourier series we have:

Thm: If $f \in L^2(I)$ and $f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n$, then

a) $\sum_{n=0}^{\infty} |c_n|^2 \leq \| f \|^2$ (Bessel)

b) $\sum_{n=0}^{\infty} |c_n|^2 = \| f \|^2$ iff $\lim_{n\to\infty} \| f - s_n \| = 0$

Of course in general, $\lim_{n\to\infty} \| f - s_n \|$ may not be zero, So even if $\sum_{n=0}^{\infty} |c_n|^2$ exists, it may not be equal to $\|f\|^2$. It is, however, equal to $\|g\|^2$ for some $g \in L^2(I)$:

<u>Thm</u>: (Riesz-Fischer) Suppose $\sum_{k=0}^{\infty} |c_k|$ converges. Then $\exists\, g \in L^2(I)$ such that

a) $(g, \varphi_k) = c_k \;\; \forall k \geq 0$, and

b) $\|g\|^2 = \sum_{k=0}^{\infty} |c_k|^2$.

Now we investigate: when does a Fourier series converge, and to what? For this we need to choose a particular orthonormal family, since convergence results can depend on the basis $\{\varphi_k\}_{k=0}^{\infty}$.

So we study

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

For this investigation, we had to prepare a bunch of lemmas/propositions surrounding the complicated integrals and limits which arise. They were:

<u>Thm</u> (Riemann-Lebesgue Lemma) If $f \in L(I)$, then

$$\lim_{\alpha \to \infty} \int_I f(t) \sin(\alpha t + \beta)\, dt = 0$$

Jordan's and Dini's conditions for the evaluation of Dirichlet integrals:

Jordan: If f is BV on $[0,\delta]$ then

$$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta g(t)\,\frac{\sin \alpha t}{t}\,dt = g(0+)$$

Dini: If $g(0+)$ exists and $\int_0^\delta \frac{g(t)-g(0+)}{t}\,dt$ exists,

then $$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta g(t)\,\frac{\sin(\alpha t)}{t}\,dt = g(0+).$$

Our first convergence result came in the form of the Riemann localization theorem. It said

Theorem (Riemann Localization) Paraphrasing:
The Fourier series of a function $f(x)$ converges at x_0 iff $\exists\,\delta$ with $0<\delta<\pi$ s.t.

$$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta \frac{f(x+t)+f(x-t)}{2}\,\frac{\sin(n+\frac{1}{2})t}{t}\,dt \quad \text{exists.}$$

Jordan's and Dini's tests can both be applied to this limit, so we found:

Theorem: If $f(x)$ satisfies the conditions of either Jordan's or Dini's test, then (ie for some x $\exists\,\delta>0$ s.t. ...) then the Fourier series converges at x to

$$\frac{f(x+)+f(x-)}{2}.$$

So if $f(x)$ is continuous at a point x_0 and satisfies either Jordan or Dini there, then the Fourier series of $f(x)$ converges to $f(x_0)$.

In fact this is true in general, for any continuous function (not just ones that additionally satisfy the hypotheses of Jordan's or Dini's theorem). The proof is a consequence of Fejér's theorem:

<u>Theorem</u>: If $f \in L([0, 2\pi])$ and is ~~continuous and~~ 2π-periodic, then the Fourier series of f $(C,1)$-sums to

$$\frac{f(x+)+f(x-)}{2}$$

whereever the above limit exists. If f is continuous it always exists, and the $(C,1)$ convergence is uniform.

This yields; since $(C,1)$ summing gives the same limit as usual summing when both limits exist:

<u>Thm</u>: If f is cts on $[0, 2\pi]$ and 2π-periodic, then its Fourier series, when it converges, converges to $f(x)$.

<u>Proof</u>: If $\{s_n(x)\}$ has a limit, its limit is the same as $\{\sigma_n(x)\}$, and $\{\sigma_n(x)\}$ always has $f(x)$ as a limit by Fejér's theorem.

Fejér's theorem also gave:

a) $\lim_{n\to\infty} \|f - s_n\| = 0$,

b) Parseval's thm holds,

c) Fourier series can be integrated term-by-term

Remark: Fourier series cannot, in general, be differentiated term-by-term.

We then moved on to study integral representations of functions, rather than representations as sums.

Thm: (Fourier integral theorem)
Assume f satisfies the hypotheses of either Jordan's or Dini's test, and $f \in L(\mathbb{R})$. Then we have cosine form:

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(u)\cos(v(u-x))\,du\right]dv \quad , \text{ and}$$

And exponential form:

$$\frac{f(x+)+f(x-)}{2} = \lim_{\alpha\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\overset{e^{iv(u-x)}}{\cancel{\cos(v(u-x))}}\,du\right]dv.$$

This can all be re-explained in terms of Fourier transforms

Note

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{iv(u-x)}\,du\,dv$$

$$= \int_{-\infty}^{\infty} e^{-ivx}\left[\int_{-\infty}^{\infty} f(u)e^{ivu}\,du\right]dv$$

Change v to $-v$

$$= \int_{-\infty}^{\infty} e^{+ivx} \left[\int_{-\infty}^{\infty} f(u)\, e^{-ivu}\, du \right] dv$$

(Note: Set $w = -v$, then $dv = -dw$ and $\int_{-\infty}^{\infty}$ becomes $\int_{\infty}^{-\infty}$, which flips back and cancels with $-dw$)

$$\frac{f(x+)+f(x-)}{2} = \int_{-\infty}^{\infty} \mathcal{F}(f)\, e^{ivx}\, dv.$$ Thus the Fourier transform, via the Fourier integral theorem, has an "inverse", because if f is continuous then the LHS is just f again.

Convolution tells us how the Fourier inversion formula behaves with respect to products (we already know it's linear):

$$\mathcal{F}^{-1}(F(s)\, G(s)) = f * g.$$

and last, Poisson summation relates the limit of the Fourier series of f to its Fourier integral:

<u>Theorem</u>: If f increases on $(-\infty, 0]$, decreases on $[0, \infty)$, is nonnegative and $\int_{-\infty}^{\infty} f(x)\, dx$ exists, then

$$\sum_{n \in \mathbb{Z}}' \frac{f(n+)+f(n-)}{2} = \sum_{n \in \mathbb{Z}}' \mathcal{F}(f)(2n\pi)$$