(2) If we take $f(x)=1$, the constant function, then $\sigma_n = s_n = 1$ for each n (compute the Fourier coeffs and check they're all 0) and we arrive at
(except the first)

$$\frac{1}{n\pi}\int_0^{\pi} \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt = 1.$$

Thus, for any $s \in \mathbb{R}$ we can write

$$\sigma_n(x) - s = \frac{1}{n\pi}\int_0^{\pi}\left(\frac{f(x+t)+f(x-t)}{2} - s\right)\frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt \quad (*)$$

therefore if we succeed in finding s such that

$$\lim_{n\to\infty}\frac{1}{n\pi}\int_0^{\pi}\left(\frac{f(x+t)+f(x-t)}{2} - s\right)\frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt = 0$$

then it follows that $\lim_{n\to\infty}\sigma_n = s$, and we'll have found the Cesàro sum! The next theorem tells us how to choose s (in a fashion depending on x).

Theorem: (Fejèr) Assume $f \in L([0,2\pi])$ and suppose that f is periodic with period 2π. Define

$$s(x) = \lim_{t\to 0^+}\frac{f(x+t)+f(x-t)}{2}$$

whenever the limit exists. Then whenever $s(x)$ is defined the Fourier series of f is $(C,1)$ summable and

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx) = s(x) \quad (C,1)$$

i.e. $\lim_{n\to\infty} \sigma_n = S_n$ $\forall x$ where $S_n := S_n(x)$ is defined.

If f is continuous on $[0, 2\pi]$ then $\{S_n(x)\}$ converges uniformly on $[0, 2\pi]$ to f.

Proof: Set $g_x(t) = \dfrac{f(x+t) + f(x-t)}{2} - s(x)$, whenever $s(x)$ is defined. Then $g_x(t) \to 0$ as $t \to 0^+$.

Thus, $\forall \varepsilon > 0$ $\exists\, \delta < \pi$ s.t. $|g_x(t)| < \frac{\varepsilon}{2}$ whenever $0 < t < \delta$.

Note that δ depends on both x and ε above, unless f is continuous on $[0, 2\pi]$ – In this case, $f(x)$ is uniformly continuous on $[0, 2\pi]$, thus so is $g_x(t)$ (thinking here of $g_x(t)$ as depending on x for a fixed t) and thus $\exists$ one δ that works for all x.

Now using the integral formula above $(*)$ we integrate $\int_0^\delta$ and $\int_\delta^\pi$:

$$\left| \frac{1}{n\pi} \int_0^\delta g_x(t) \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)} dt \right| \le \frac{\varepsilon}{2n\pi} \int_0^\delta \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)} dt = \frac{\varepsilon}{2}$$

On $[\delta, \pi]$ we get

$$\left| \frac{1}{n\pi} \int_\delta^\pi g_x(t) \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)} dt \right| \le \frac{1}{n\pi \sin^2 \frac{1}{2}\delta} \int_\delta^\pi |g_x(t)| \le \frac{I(x)}{n\pi \sin^2 \frac{1}{2}\delta}.$$

Here, we used $\left| \dfrac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)} \right| \le \dfrac{1}{|\sin^2(\frac{1}{2}t)|}$ and $\sin^2(\frac{1}{2}t)$ attains its max at π, so over $[\delta, \pi]$ it attains its min at $x = \delta$, thus

$$\frac{1}{|\sin^2(\frac{1}{2}t)|} \le \frac{1}{\sin^2(\frac{\delta}{2})} \quad \text{over } t \in [\delta, \pi]. \quad \text{Here,}$$

$I(x) = \int_0^\pi |g_x(t)|\,dt$. Choose N s.t. $\dfrac{I(x)}{N\pi\sin^2(\frac{\delta}{2})} < \dfrac{\varepsilon}{2}$.

Then $\forall n \ge N$

$$|\sigma_n(x) - s(x)| = \left| \frac{1}{n\pi} \int_0^\pi g_x(t) \frac{\sin^2(\frac{1}{2}nt)}{\sin^2(\frac{1}{2}t)}\,dt \right| < \varepsilon,$$

so $\lim_{n\to\infty} \sigma_n(x) = s(x)$.

For uniform convergence, if f is continuous on $[0, 2\pi]$ and 2π-periodic then f is bounded on $\mathbb{R}$. Then $g_x(t)$ is similarly bounded, and $\exists\, M$ s.t. $|g_x(t)| \le M \;\forall x, t$. Then $I(x)$ is replaced by πM above, and the resulting N is chosen so that

$$\frac{M}{N\sin^2(\frac{\delta}{2})} < \frac{\varepsilon}{2},$$

ie. it's independent of x. We conclude $\sigma_n \longrightarrow s = f$ uniformly on $[0, 2\pi]$.

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§11.14 Consequences of Fejér's Theorem.

First, a note: Several times last class I wrote $\frac{f(x+t)-f(x-t)}{2}$ instead of $\frac{f(x+t)+f(x-t)}{2}$, the latter is correct. Please be sure my error did not make it into your notes.

Theorem 11.16: Let f be continuous on $[0, 2\pi]$, and 2π-periodic. Let S_n denote the usual partial sums, and

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Then

a) $\lim_{n\to\infty} S_n = f$ on $[0, 2\pi]$, ie $\lim_{n\to\infty} \| f - s_n \| = 0$

b) $\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2$ (Parseval's)

c) The Fourier series can be integrated term-by-term:

$$\int_0^x f(t)\,dt = \frac{a_0 x}{2} + \sum_{n=1}^{\infty} \int_0^x a_n \cos nt + b_n \sin nt \, dt.$$

Moreover the integrated series is uniformly convergent on every interval, even if the Fourier series diverges.

d) If the Fourier series converges for some x, then it converges to $f(x)$.

Proof a): Recall that if $S_n(x)$ denotes the partial sum of the Fourier series and $t_n(x)$ denotes any other sum of the first n elements of the orthonormal system (with arbitrary weights) then

$$\|f - s_n\| \le \|f - t_n\|$$

So set $t_n = \sigma_n = \frac{1}{n}\sum_{k=0}^{n-1} S_k$, then

$$\int_0^{2\pi} |f(x) - s_n(x)|^2 dx \le \int_0^{2\pi} |f(x) - \sigma_n(x)|^2 dx.$$

However $\sigma_n \to f$ uniformly on $[0, 2\pi]$, so by Fejér's Thm:

$$\lim_{n\to\infty} \|f - s_n\| = \lim_{n\to\infty} \int_0^{2\pi} |f(x) - S_n(x)|^2 dx$$

$$\le \lim_{n\to\infty} \int_0^{2\pi} |f(x) - \sigma_n(x)|^2 dx$$

$$= \int_0^{2\pi} \lim_{n\to\infty} |f(x) - \sigma_n(x)|^2 dx = 0.$$

b) Part b is now a consequence of earlier calculations, since the hypothesis $\lim_{n\to\infty} \|f - s_n\| = 0$ of Fejér's theorem is satisfied.

c) This is a consequence of:

Theorem 9.18: Assume $\lim_{n\to\infty} \|f_n - f\| = 0$ on $[a, b]$, $g \in R$ on $[a, b]$ and set $h(x) = \int_a^x f(t) g(t) dt$, $h_n(x) = \int_a^x f_n(t) g(t) dt$, $x \in [a, b]$.

Then $h_n \longrightarrow h$ uniformly on $[a, b]$.

So from (a) we get that (with $g(x)=1$) that:

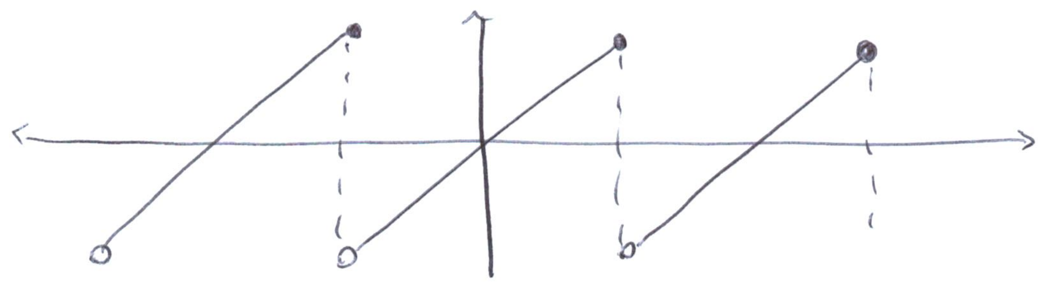
$\int_a^x S_n(t)\,dt$ converges uniformly to $\int_a^x f(t)\,dt$

Thus we can integrate term-by-term.

d) We already know $\sigma_n(x) \longrightarrow f(x)$. If $S_n(x)$ converges, then $\sigma_n(x)$ converges to the same number. So if the Fourier series converges, it converges to $f(x)$.

Remarks: (a) implies (b) and (c) and can be proved without continuity. (d) requires continuity for the limits to behave as claimed. Also (a) holds for all $f \in L^2([0,2\pi])$.

Example: Suppose $f(x)=x$, $-\pi \le x \le \pi$, extended to be 2π periodic. Then



If $f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ then we first conclude $a_k = 0$ since f is odd. On the other hand

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin kt\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t \sin kt\,dt = \frac{2}{\pi} t \frac{-\cos kt}{k}\Big|_0^{\pi} + \frac{2}{\pi}\int_0^{\pi} 1.\, \frac{\cos kt}{k}\,dt$$

$$= \frac{2}{\pi}\pi \cdot \frac{(-\cos k\pi)}{k} + \underbrace{\frac{2}{\pi k}\,\frac{\sin kt}{k}\Big|_0^{\pi}}_{=0,\ \sin 0 = 0,\ \sin k\pi = 0}$$

$= \frac{2}{n}(-1)^{n+1}$. Therefore

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx. \text{ Then}$$

$$\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx = \lim_{t \to 0^+} \frac{f(x+t) + f(x-t)}{2} = \begin{cases} x & \text{if } -\pi < x < \pi \\ 0 & \text{if } x = \pi. \end{cases}$$

Then e.g. if $x = \frac{\pi}{2}$ $\sin(k\pi/2) = \underset{k=1}{1}, 0, -1, 0, 1, 0, -1, \ldots$

$$\sin \frac{k\pi}{2} = \begin{cases} 0 & \text{if } k = 2n \\ (-1)^n & \text{if } k = 2n+1 \end{cases} \quad n \in \mathbb{Z}.$$

So if we set $x = \pi/2$ then

$$\frac{\pi}{2} = \lim_{t \to 0^+} \frac{f(x+t) + f(x-t)}{2} = \sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+1}$$

In particular we get the famous formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Note that in this case, we know that while the Fourier series of f <u>may</u> converge pointwise to f, it cannot converge uniformly to $f(x)$:

The uniform limit of the $s_n(x)$ must be continuous since $s_n(x)$ are continuous. However the function $f(x)$ is not cts.

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We've just seen a consequence of Fejer's theorem:

11.16 d): If $f(x)$ is cts on $[0, 2\pi]$ and 2π-periodic, then if the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

converges for some x, then it converges to $f(x)$.

And we would like to combine this with either Dini's or Jordan's test to say that the Fourier series of f converges to f. I.e.

Jordan's Test: If f is BV on some compact interval $[x-\delta, x+\delta]$ for some $\delta < \pi$, then the Fourier series converges to

$$S(x) = \lim_{t \to 0^+} \frac{f(x+t) + f(x-t)}{2} \quad \text{at } x.$$

However these two facts do not combine neatly: There are certainly BV functions that are not cts, but in fact there are also cts functions that are not BV:

Idea: Use

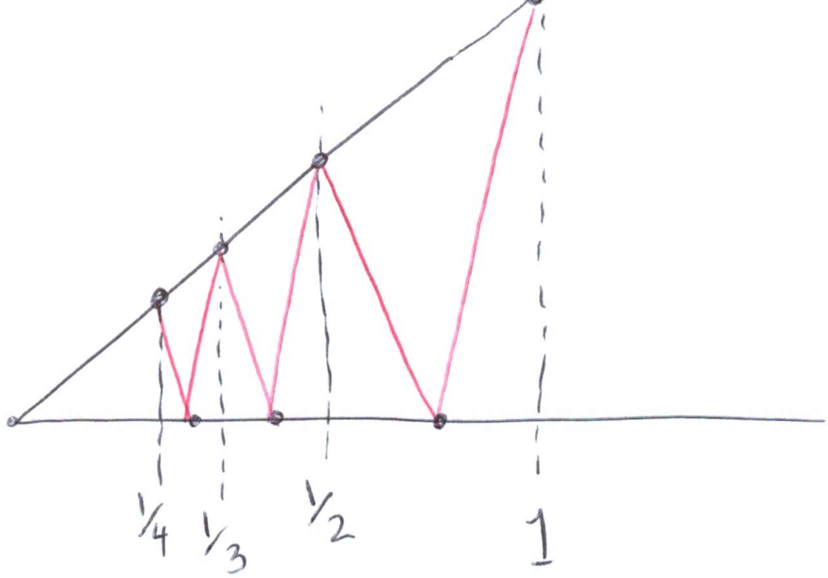
$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\frac{1}{x} & \text{if } x > 0 \end{cases} \quad \text{on } [0, 1].$$

To make a function that's easier to work with, we do:

Define $f(\frac{1}{n}) = \frac{1}{n}$, $n \in \mathbb{N}$ and

$$f\left(\frac{\frac{1}{n} + \frac{1}{n+1}}{2}\right) = 0.$$

Then connect these points linearly, we get



and also set $f(0)=0$. Then f is continuous, but on $[\frac{1}{n},1]$ $\mathrm{Var} f$ is $1+\underbrace{2\sum_{k=2}^{n-1}\frac{1}{k}}+\frac{1}{n}$

This counts all the "ups and downs" to each $1/k$.

So $\lim_{n\to\infty} \mathrm{Var} f$ on $[\frac{1}{n},1] = \lim_{n\to\infty} 1+2\sum_{k=2}^{n-1}\frac{1}{k}+\frac{1}{n} = \infty$,

so f is not BV.

Similarly, Dini's test was (same assumptions on f):

If $\lim_{t\to 0^+} \frac{f(x+t)+f(x-t)}{2} = s(x)$ exists and if $\int_0^\delta \frac{g(t)-s(x)}{t}\,dt$ exists for some $\delta<\pi$, then the Fourier series of f converges to $s(x)$ at x. Here $g(t) = \frac{f(x+t)+f(x-t)}{2}$.

Does not "pair well" with continuity.

Exercise: Dini's hypotheses do not follow from assuming f is continuous. Similarly f cts does not imply Dini's hypotheses hold.

Example: A continuous function with a divergent Fourier series (due to Fejèr).

Lemma: Set

$$\phi(n,r,x) = \frac{\cos(r+1)x}{2n-1} + \frac{\cos(r+2)x}{2n-3} + \cdots + \frac{\cos(r+n)x}{1}$$

$$- \frac{\cos(r+n+1)x}{1} - \frac{\cos(r+n+2)x}{3} - \cdots - \frac{\cos(r+2n)x}{2n-1}.$$

Then $\phi(n,r,x)$ is bounded for all n, r, x.

Proof: A long series of trig identities.

Now set, for $n \in \mathbb{Z}$,

$$G_n = \left\{ \frac{1}{2n-1}, \frac{1}{2n-3}, \ldots, \frac{1}{3}, 1, -1, -\frac{1}{3}, \ldots, -\frac{1}{2n-1} \right\}$$

Let $\lambda_i \in \mathbb{Z}$ be a sequence ~~with~~ of strictly increasing integers. Consider the collections $G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \ldots$ etc of numbers. Multiply the elements of each G_{λ_i} by i^{-2}, to obtain collections:

$\frac{1}{1^2} \cdot \frac{1}{2\lambda_1 - 1}, \ldots, \frac{-1}{1^2} \cdot \frac{1}{2\lambda_1 - 1}$ ← what G_{λ_1} became

$\frac{1}{2^2} \cdot \frac{1}{2\lambda_2 - 1}, \ldots, \frac{-1}{2^2} \cdot \frac{1}{2\lambda_2 - 1}$ ← what G_{λ_2} became

⋮

etc

Listing these elements in order as above, rename the resulting sequence $\alpha_1, \alpha_2, \ldots$ e.g.

$$\alpha_1 = \frac{1}{1^2(2\lambda_1 - 1)}, \quad \alpha_2 = \frac{1}{1^2(2\lambda_1 - 3)}, \ldots, \alpha_{2\lambda_1} = \frac{-1}{1^2(2\lambda_1 - 1)}, \quad \alpha_{2\lambda_1+1} = \frac{1}{2^2(2\lambda_2 - 1)}$$

... etc

Consider $\sum_{n=1}^{\infty} \alpha_n \cos nx$.

Suppose we take all of the terms corresponding to a collection G_{λ_i} and group them together in brackets. Then the bracketed sum is

$$\sum_{n=1}^{\infty} \frac{\phi(\lambda_n, 2\lambda_1 + 2\lambda_2 + \dots + 2\lambda_{n-1}, x)}{n^2}.$$

(Just write out the formulas and check). By our lemma, $\phi(n, r, x)$ is bounded for all n, r, x, and so this series is absolutely and uniformly convergent. Set

$$f(x) = \sum_{n=1}^{\infty} \frac{\phi(\lambda_n, 2\lambda_1 + \dots + 2\lambda_{n-1}, x)}{n^2}.$$

Then $f(x)$ is continuous. Note that this is <u>not</u> a rearrangement of $\sum_{n=1}^{\infty} \alpha \cos nx$, so we've got something genuinely new.

E.g. If $\sum_{k=1}^{\infty} (-1)^k = -1+1-1+1-1+1\dots$

Then $\sum_{k=1}^{\infty} (-1+1) = \sum_{k=1}^{\infty} 0 = 0$ is not a rearrangement,

a rearrangement is: If $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection, then $\sum_{i=1}^{\infty} a_{f(i)}$ is a rearrangement of $\sum_{i=1}^{\infty} a_i$.

<u>However</u>, it turns out that $\sum_{n=1}^{\infty} \alpha \cos nx$ is the Fourier series of our function $f(x)$!

To see this, observe that since $f(x)$ is defined by an absolutely and uniformly convergent series, we can compute:

$$\int_0^{2\pi} f(x)\sin(mx)\,dx = \int_0^{2\pi}\sum_{n=1}^{\infty}\frac{\phi(\lambda_n, 2\lambda_1+\dots+2\lambda_{n-1}, x)}{n^2}\sin(mx)\,dx$$

$$= \sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\phi(\lambda_n, 2\lambda_1+\dots+2\lambda_{n-1}, x)}{n^2}\sin(mx)\,dx$$

$$= 0, \quad \text{since } \int_0^{2\pi}\sin(mx)\cos(nx)\,dx = 0 \ \forall n,m\in\mathbb{Z}$$

Whereas

$$\int_0^{2\pi} f(x)\cos(mx)\,dx = \sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\phi(\lambda_n, 2\lambda_1+\dots+2\lambda_{n-1}, x)}{n^2}\cos(mx)\,dx$$

$= \pi\alpha_m$. This is because every integral $\int_0^{2\pi}\cos(mx)\cos(nx)\,dx$ with $m\neq n$ gives zero, while

$$\int_0^{2\pi}\alpha_m\cos(mx)\,dx = \pi\alpha_m.$$

Thus the numbers α_m are the Fourier cosine coeffs of $f(x)$.

Finally we return to the numbers λ_i and show that they can be chosen so that at $x=0$, the Fourier series diverges. That is,

$$\sum_{n=1}^{\infty}{}' \alpha_n\cos(n\cdot 0) = \sum_{n=1}^{\infty}\alpha_n \text{ diverges.}$$

Let $s_n = \sum_{k=1}^{n} \alpha_k$. Then

$$s_{2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{i-1} + \lambda_i} = \frac{1}{i^2}\left(\frac{1}{2\lambda_i - 1} + \frac{1}{2\lambda_i - 3} + \cdots + \frac{1}{3} + 1\right)$$

(because $s_{2\lambda_1} = 0$, $s_{2\lambda_1 + 2\lambda_2} = 0$, etc.)

However, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is asymptotic to $\ln(n)$, ie

$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \ln(n)\right) = 0$. Thus

$$s_{2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{i-1} + \lambda_i} \sim \frac{\ln(\lambda_i)}{i^2}.$$

If we choose $\{\lambda_i\}$ so that $\lambda_i \longrightarrow \infty$ very quickly, then $\frac{\ln(\lambda_i)}{i^2}$ will diverge (thus the partial sums of $\sum_{k=1} \alpha_k$ diverge). E.g. if we set

$\lambda_i = i^{i^2}$, then

$$s_{2\lambda_1 + \cdots + 2\lambda_{i-1} + \lambda_i} \sim \frac{\ln(i^{i^2})}{i^2} = \frac{i^2 \ln(i)}{i^2} = \ln(i),$$

so $\lim_{n\to\infty} s_n$ does not exist, thus $\sum_{k=1}^{\infty} a_k$ does not converge.

Thus $f(x)$, while continuous everywhere, does not have a convergent Fourier series at $x=0$.