MATH 3472

307-367

Today, we begin multivariable calculus. That is, calculus of functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$, things like

$$f(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n), f_2(x_1,\ldots,x_n), \ldots, f_m(x_1,\ldots,x_n)).$$

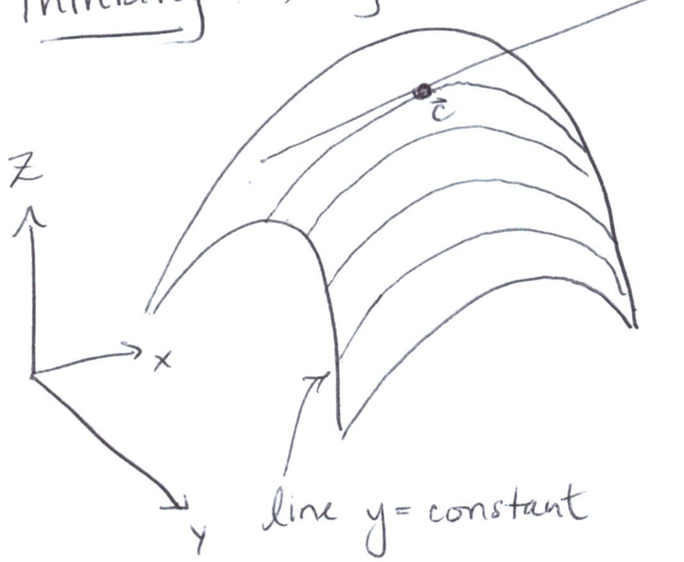
You've already seen one special case of this (maybe more?) in the form of functions $f:\mathbb{R}^n \longrightarrow \mathbb{R}$. There, we considered <u>partial derivatives</u> defined as follows:

<u>Def</u>: Let $S \subset \mathbb{R}^n$ be an open set, and $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ a function. If $\vec{x} = (x_1,\ldots,x_n)$ and $\vec{c} = (c_1,\ldots,c_n)$ are points in S with $x_i = c_i$ for $i \neq k$ and $x_k \neq c_k$, then

$$D_k f(\vec{c}) = f_k(\vec{c}) = \frac{\partial f}{\partial x_k}(\vec{c}) = \lim_{x_k \to c_k} \frac{f(\vec{x}) - f(\vec{c})}{x_k - c_k}$$

is the k-the partial derivative of $f(x)$.

<u>Thinking</u>: $f(x,y)$



the slope of this line would be the number

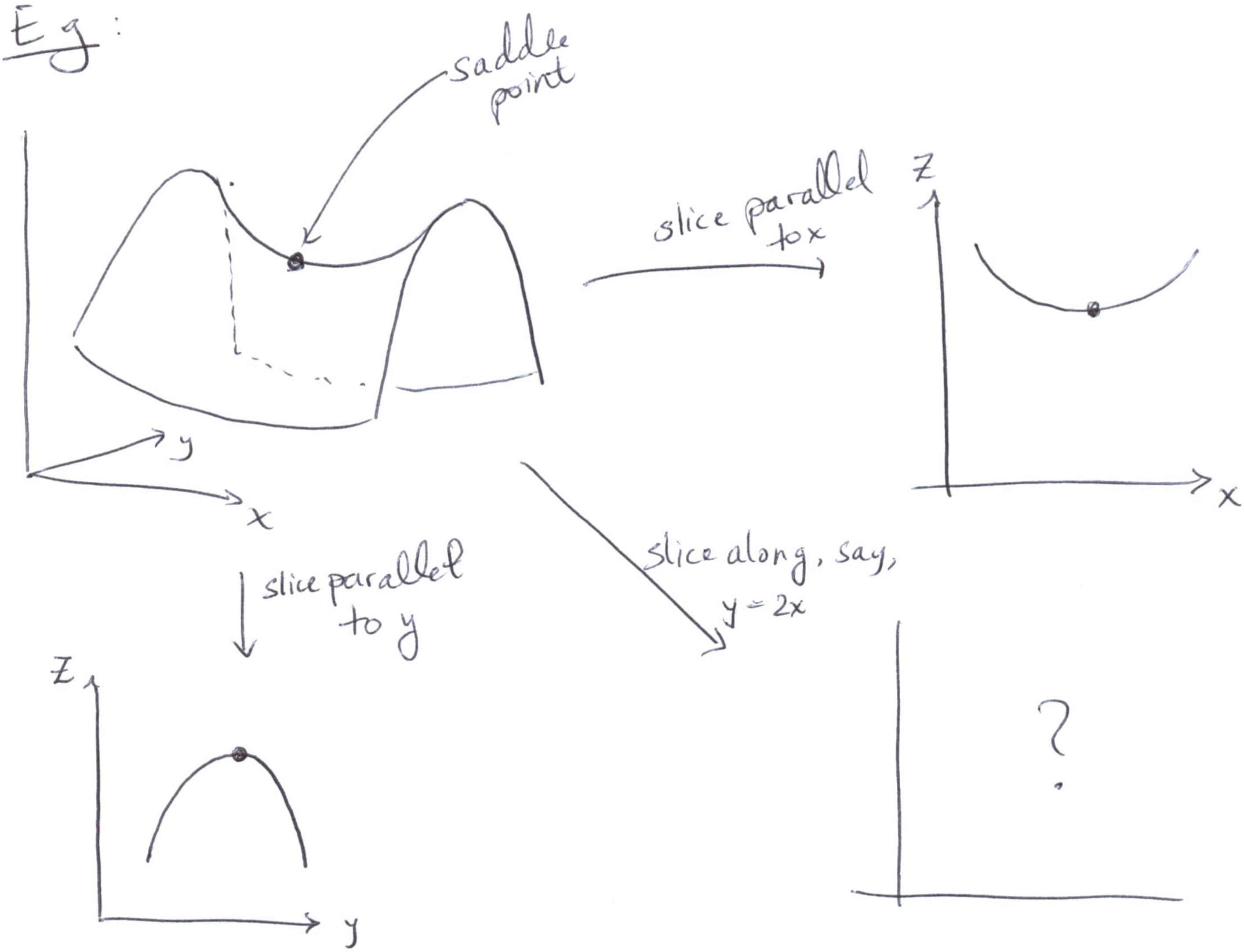
$$\frac{\partial f}{\partial x}(\vec{c}).$$

The partial derivatives were inadequate because:

- Existence of all partials at $\vec{x} = \vec{c}$ does not imply continuity of f at $\vec{c}$.
- Only gives information about the rate of change of f in the direction of a coordinate axis

<u>Natural improvement</u>: Instead of $\dfrac{\partial f}{\partial x_i}$, which is "slicing" the graph of f parallel to the x_i-axis, why not allow a "slice" in any direction and consider the derivative of resulting curves?

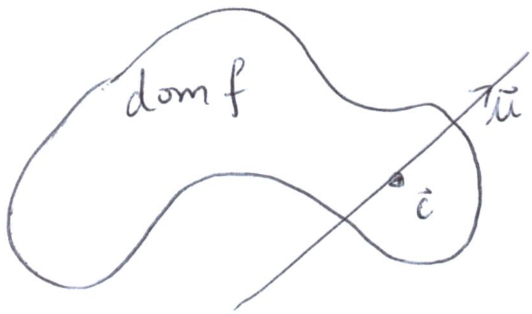
<u>Eg</u>:



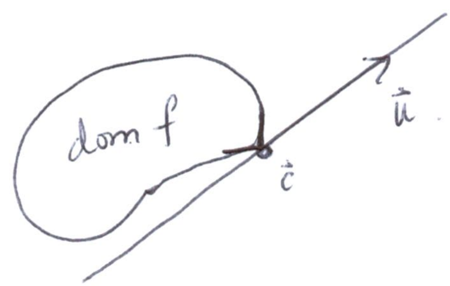
The directional derivative is the solution to this problem.

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be given, and $S \subset \mathbb{R}^n$ a set contained in the domain of f. Let $\vec{c} \in S$ be given, and choose any vector $\vec{u} \in \mathbb{R}^n$. We want to study how f changes as we move along the line $\vec{c} + t\vec{u}$, $t \in \mathbb{R}$. (So in our previous picture, we move from $\vec{c} = (0,0)$ along the line $(t, 2t)$).

One small problem: There has to be a small section of line $\vec{c} + t\vec{u}$, passing through $\vec{c}$, which lies in the domain of f. E.g we need:



and not



So we insist $c \in \operatorname{int}(S)$. Then $\exists$ an n-ball $B(\vec{c}, r)$ $(r > 0)$ s.t. $B(\vec{c}, r) \subset S$, so the segment of $\vec{c} + t\vec{u}$ inside $B(\vec{c}, r)$ is in S.

Now define:

<u>Definition</u>: The directional derivative of f at $\vec{c}$ in the direction of $\vec{u}$ is

$$f'(\vec{c};\vec{u}) = \lim_{h\to 0} \frac{f(\vec{c}+h\vec{u})-f(\vec{c})}{h}$$

when the limit exists.

<u>Rmk</u>: We don't require $\|\vec{u}\|=1$, but some authors do.

<u>Examples/Notes</u>:

(i) If $\vec{u}=\vec{0}$ then the definition still makes sense.

$$f'(\vec{c};\vec{u}) = \vec{0} \quad \text{for all } f, \vec{c}\in S.$$

(ii) If $\vec{u} = (0,0,\ldots,1,\ldots,0)$ and $m=1$,

$$\underset{\substack{\uparrow \\ 1 \text{ in } k^{th} \text{ position}}}{}$$

then $f'(\vec{c},\vec{u}) = \frac{\partial f}{\partial x_k}(\vec{c})$, and in general when $m\neq 1$ we take this as the <u>definition</u> of the k^{th} partial derivative of $f:\mathbb{R}^n\to\mathbb{R}^m$. That is, the k^{th} partial of $f:\mathbb{R}^n\to\mathbb{R}^m$ is its directional derivative in the direction of the k^{th} unit vector.

(iii) If $f=(f_1,\ldots,f_m)$ then $f'(\vec{c};\vec{u})$ exists if and only if $f_i'(\vec{c};\vec{u})$ exists for $i=1,\ldots,m$.

(This follows from the properties of limits), and we get

$$f'(\vec{c};u) = (f_1'(\vec{c};\vec{u}), f_2'(\vec{c},\vec{u}), \ldots, f_n'(\vec{c},\vec{u}))$$

For partials, this means

$$D_k f'(\vec{c}) = (D_k f_1(\vec{c}), \ldots, D_k f_m(\vec{c})).$$

Since partial derivatives of functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ are computed by fixing all coordinates x_i of a function $f(x_1, \ldots, x_n)$ except for one, they are computed by simply treating all variables as constants, except for one. The proof of this fact is a straightforward check. Thus, e.g.

$$\frac{\partial}{\partial x_2}\left(x_1 \cos x_2 + \ln(x_3)\right) = -x_1 \sin(x_2),$$

$$\frac{\partial}{\partial x_3}\left(x_1 \cos x_2 + \ln(x_3)\right) = \frac{1}{x_3}$$

$$\frac{\partial}{\partial x_1}\left(x_1 \cos(x_2) + \ln(x_3)\right) = \cos(x_2).$$

Since partials are computable in this way, it's very good to express other derivatives in terms of partials.

A special case of a general theorem we'll eventually prove deserves mention here. If $x_i : \mathbb{R} \to \mathbb{R}$ are real functions, and $f : \mathbb{R}^n \to \mathbb{R}$ is a function, then

$g(t) = f(x_1(t), x_2(t), \dots, x_n(t))$ is a real-valued function of one variable. Its derivative is found by a generalized chain rule, it is:

$$g'(t) = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt}.$$

For directional derivatives of functions $f : \mathbb{R}^n \to \mathbb{R}$, this gives:

<u>Example</u>: Given $\vec{c} = (c_1, \dots, c_n)$ in the interior of the domain of $f ; \mathbb{R}^n \to \mathbb{R}$, and $\vec{u} \in \mathbb{R}^n$ any direction, set $x_i(t) = c_i + tu_i$ for all $i = 1, \dots, n$.

Then the directional derivative of f in the direction of $\vec{u}$ at $\vec{c}$ is

$$f'(\vec{c};\vec{u}) = \lim_{h \to 0} \frac{f(c_1 + hu_1, c_2 + hu_2, \dots, c_n + hu_n) - f(c_1, c_2, \dots, c_n)}{h}$$

$= g'(0)$. So from the chain rule above;

when $f : \mathbb{R}^n \to \mathbb{R}$ we can calculate

$$f'(\vec{c},\vec{u}) = g'(0) = \underbrace{u_1}_{\frac{dx_1}{dt}} \frac{\partial f}{\partial x_1}(0) + \underbrace{u_2}_{\frac{dx_2}{dt}} \frac{\partial f}{\partial x_2}(0) + \dots + \underbrace{u_n}_{\frac{dx_n}{dt}} \frac{\partial f}{\partial x_n}(0)$$

Example: If $f(x,y,z) = \cos(xyz) + xyz + 2x$ then $f'(\vec{0};\vec{u})$ when $u = [1,2,3]$ is

$$1 \cdot \frac{\partial f}{\partial x}(0) + 2 \cdot \frac{\partial f}{\partial y}(0) + 3 \cdot \frac{\partial f}{\partial z}(0)$$

$$= 1 \cdot \overbrace{\left(yz(-\sin(xyz)) + yz + 2\right)\Big|_{(x,y,z)=(0,0,0)}}^{=2}$$

$$+ 2 \cdot \overbrace{\left(xz(-\sin(xyz)) + xz\right)\Big|_{(x,y,z)=(0,0,0)}}^{=0}$$

$$+ 3 \cdot \overbrace{\left(xy(-\sin(xyz)) + xy\right)\Big|_{(x,y,z)=(0,0,0)}}^{=0}$$

$$= 1 \cdot 2 + 2 \cdot 0 + 3 \cdot 0 = 2.$$

Example: Linear functions. If $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$, then

$$\lim_{h\to 0} \frac{f(\vec{c}+h\vec{u}) - f(\vec{c})}{h} = \lim_{h\to 0} \frac{f(h\vec{u})}{h} = \lim_{h\to 0} \frac{hf(\vec{u})}{h}$$

$$= f(\vec{u}).$$

Though we can see it from the definition, this is perhaps a good time to point out:

Since we don't require $\|\vec{u}\|=1$, our directional derivatives depend on both the direction and magnitude of $\vec{u}$.

E.g. in our previous example with $\vec{u}=(1,2,3)$ we found

$$f'(\vec{0};\vec{u})=2$$

while $f'(\vec{0};2\vec{u})=4$, for example. This is true in general since

$$f'(\vec{c};k\vec{u})=\lim_{h\to 0}\frac{f(\vec{c}+hk\vec{u})-f(\vec{c})}{h}$$ depends on k.

Another way of saying this is:

If $F(t) = f(\vec{c}+t\vec{u})$, then $F'(0) = f'(\vec{c};\vec{u})$, and in general

$$F'(t) = f'(c + t\vec{u}\,;\vec{u})$$ when either derivative exists.

Example: If $f(x) = \|x\|^2$, then

dot product

$$F(t) = f(\vec{c}+t\vec{u}) = (\vec{c}+t\vec{u})\cdot(\vec{c}+t\vec{u})$$

$$= \|\vec{c}\|^2 + 2t\,\vec{c}\cdot\vec{u} + \|\vec{u}\|^2 t^2$$

$$\Rightarrow F'(t) = 2\vec{c}\cdot\vec{u} + 2t\|\vec{u}\|^2,$$ in particular

$$f'(\vec{c};\vec{u}) = F'(0) = 2\vec{c}\cdot\vec{u}.$$

§12.3.

Relationships between partial and directional derivatives, continuity

First, fix $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $c \in$ interior of domain of f.

Since partials $\frac{\partial f}{\partial x_k}(\vec{c})$ are all special cases of directional derivatives, we have:

Fact: If all directional derivatives $f'(\vec{c};\vec{u})$ exist, then all partials $\frac{\partial f}{\partial x_k}(\vec{c})$ exist as well.

The converse is not true. Again consider

$$f(x,y) = \begin{cases} x+y & \text{if } x \text{ or } y \text{ is } 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then $f: \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 1$,

since (for example)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h+0 - 0+0}{h} = 1.$$

On the other hand, choose any $\vec{u} = (u_1, u_2)$ where $u_1 \neq 0$ and $u_2 \neq 0$. Then

$$f'((0,0);\vec{u}) = \lim_{h\to 0} \frac{f((0,0)+h(u_1,u_2)) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1-0}{h} = \lim_{h\to 0} \frac{1}{h}, \text{ does not exist.}$$

We also already saw that a function f can have all partials at some point $\vec{c}$, yet fail to be continuous there (in fact, f above works).

<u>Claim</u> A function f can have all directional derivatives $f'(\vec{c};\vec{u})$, yet fail to be continuous at $\vec{c}$.

<u>Proof of claim</u>:

Let $f(x,y) = \begin{cases} x^2y/(x^4+y^2) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$

Then let $\vec{u} = (u_1, u_2)$ be any vector, and $\vec{c} = (0,0)$.

We compute, if $u_2 \neq 0$, that:

$$\lim_{h\to 0} \frac{f(\vec{0}+h\vec{u}) - f(\vec{0})}{h} = \lim_{h\to 0} \frac{h^3 u_1^2 u_2}{h^4 u_1^4 + h^2 u_2^2} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{u_1^2 u_2}{u_2^2 + \cancel{h^2 u_1^4}} = \frac{u_1^2}{u_2}$$

and if $u_2 = 0$ then

$$\lim_{h\to 0} \frac{f(\vec{0}+h\vec{u}) - f(\vec{0})}{h} = 0, \text{ so}$$

$$f'(\vec{c}; u) = \begin{cases} \frac{u_1^2}{u_2} & \text{if } u_2 \neq 0 \\ 0 & \text{if } u_2 = 0. \end{cases}$$, so all directional derivatives exist.

On the other hand, f is not continuous because for any $t \in \mathbb{R}, t \neq 0$

$$f(t, t^2) = \frac{t^2 \cdot t^2}{t^4 + t^4} = \frac{1}{2}$$

while if $t = 0$ then

$$f(0,0) = 0.$$

So the behaviour of f along the parabola (t, t^2) is discontinuous.

(previous class' notes)

§12.4 Total derivative

For a function $f: \mathbb{R} \longrightarrow \mathbb{R}$, the derivative at $x_0 \in (a,b)$ exists if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{exists.}$$

Equivalently, we can rewrite this as

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0$$

and further rearrange to get

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - \overset{m}{f'(x_0)}(x - x_0)|}{|x - x_0|} = 0.$$

Thinking of $f'(x_0) = m$ as the slope of the tangent line, we can think of "multiplication by m" as the linear function that best approximates f at x_0.

By "best approximates" we mean the equation above holds.

We generalize to higher dimensions as follows:

Def 1 (Marsden & Hoffman) A map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be differentiable at $c \in S \subset \operatorname{dom} f$ if there exists a linear map $T_c: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{x \to c} \frac{\| f(x) - f(c) - T_c(x-c) \|}{\| x - c \|} = 0.$$

Here's another approach to the same situation: Let $E_c(v)$ denote the "error" between our approximating linear function and f, so

correct for the fact that linear maps send 0 to 0

$$E_c(v) = \frac{f(\vec{c}+\vec{v}) - f(\vec{c}) - T_c(\vec{v})}{\|\vec{v}\|}$$

normalized by $\|\vec{v}\|$.

Then the derivative is:

Def (Apostol) A function f is said to be differentiable at $\vec{c}$ if there exists a linear function $T_c : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(\vec{c}+\vec{v}) = f(\vec{c}) + T_c(\vec{v}) + \|\vec{v}\| E_c(\vec{v}) \qquad (*)$$

with $\lim_{v \to 0} E_c(\vec{v}) = 0$.

As long as you understand $E_c(\vec{v})$ to be the "error term" as given above, these definitions are easily seen to be equivalent.

$(*)$ above is like a Taylor formula (first order) for functions $f: \mathbb{R}^n \to \mathbb{R}^m$, that is how the generalization from $f: \mathbb{R} \to \mathbb{R}$ to $f: \mathbb{R}^n \to \mathbb{R}^m$ is approached in Apostol.

Theorem 12.3: Assume f is differentiable at c with total derivative T_c. Then the directional derivative $f'(\vec{c}; \vec{u})$ exists for all $\vec{u} \in \mathbb{R}^n$, and

$$T_c(\vec{u}) = f'(c;u).$$

In particular, this shows that the total derivative is unique since its action on vectors $\vec{u}$ is given by $f'(\vec{c};\vec{u})$, which is unique.

<u>Proof</u>: If $\vec{u}=0$ then $f'(\vec{c};\vec{u})=0$ and $T_c(0)=0$, so the formulas agree.

If $\vec{u}\neq 0$ then consider $h\vec{u}$ in the formula $(*)$ and get:

$$f(\vec{c}+h\vec{u}) - f(\vec{c}) = T_c(h\vec{u}) + \|h\vec{u}\| E_c(h\vec{u})$$
$$= hT_c(\vec{u}) + h\|u\| E_c(h\vec{u}).$$

Therefore

$$f'(\vec{c},\vec{u}) = \lim_{h\to 0} \frac{f(c+h\vec{u}) - f(\vec{c})}{h} = \lim_{h\to 0} T_c(\vec{u}) + \lim_{h\to 0} \|u\| E_c(h\vec{u})$$

$$= T_c(\vec{u}) + \|u\| \underbrace{\cancel{\lim_{h\to 0} E_c(h\vec{u})}}_{\to 0}$$

$$= T_c(u).$$

So the formula holds.

<u>Def</u>: A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $c \in \operatorname{int}(\operatorname{dom} f)$ if T_c exists satisfying either definition above

<u>Theorem</u> 12.4: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at c, then f is continuous at c.

<u>Proof</u>: Consider the formula

$$f(\vec{c}+\vec{v}) = f(\vec{c}) + T_c(\vec{v}) + \|\vec{v}\| E_c(\vec{v}).$$

Taking limits,

$$\lim_{\vec{v}\to\vec{0}} f(\vec{c}+\vec{v}) = f(\vec{c}) + \underbrace{\lim_{\vec{v}\to\vec{0}} T_c(\vec{v})}_{\to 0} + \underbrace{\lim_{\vec{v}\to\vec{0}} \|\vec{v}\| E_c(\vec{v})}_{\to 0}$$

goes to zero since linear functions are continuous and $T_c(\vec{0}) = \vec{0}$

goes to zero since $E_c(\vec{v}) \longrightarrow 0$ and $\|\vec{v}\| \longrightarrow 0$

and therefore $\lim_{\vec{v}\to\vec{0}} f(\vec{c}+\vec{v}) = f(\vec{c})$, which is the definition of continuity. [Recall f is cts at c iff $\lim_{x\to c} f(x) = f(c)$].

<u>Remark</u>: You will also see the total derivative of $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at c written as $f'(\vec{c})$, however we must remember that this is a <u>linear function</u>. Thus $f'(\vec{c}): \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function accepting n-vectors $\vec{v} \in \mathbb{R}^n$ as its argument and we are forced to write

$$f'(\vec{c})(\vec{v})$$

to "plug in" an argument.

(Although we could write $f'(\vec{c};\vec{v})$ in place of $f'(\vec{c})(\vec{v})$, by Theorem 12.3).

Example: Suppose $f:\mathbb{R}^n \to \mathbb{R}^m$ is a linear function. Then $f(\vec{c}+\vec{v}) = f(\vec{c})+f(\vec{v})$, and if we take $T_c = f$ $\forall c$ then

$$E_c(\vec{v}) = \frac{f(\vec{c}+\vec{v}) - f(\vec{c}) - (\overbrace{f(\vec{v})}^{T_c(v)})}{\|\vec{v}\|} = \vec{0},$$

and so $E_c(\vec{v}) \to 0$ as $\vec{v} \to 0$, as required, and the equation

$$f(\vec{c}+\vec{v}) = f(\vec{c}) + T_c(\vec{v}) + \|\vec{v}\| E_c(\vec{v})$$

certainly holds.

Thus when f is a linear function, $T_c = f$ $\forall c \in \mathbb{R}^n$. (This is in ~~agg~~ agreement with $y = mx+b \Rightarrow \frac{dy}{dx} = m$, interpreted as a linear map $\mathbb{R} \to \mathbb{R}$).

* Start here *

Example: We saw last day that if $f:\mathbb{R}^n \to \mathbb{R}$ and $\vec{c}, \vec{u} \in \mathbb{R}^n$ then if $\vec{u} = (u_1, \ldots, u_n)$ we have

$$f'(\vec{c};\vec{u}) = u_1 \frac{\partial f}{\partial x_1}(\vec{c}) + u_2 \frac{\partial f}{\partial x_2}(\vec{c}) + \ldots + u_n \frac{\partial f}{\partial x_n}(\vec{c}).$$

Since $T_c(\vec{v}) = f'(\vec{c};\vec{u})$ by Theorem 12.3, we know a formula for T_c in this case: $T_c : \mathbb{R}^n \to \mathbb{R}$ is a linear function with

$$T_c(\vec{u}) = u_1 \frac{\partial f}{\partial x_1}(\vec{c}) + u_2 \frac{\partial f}{\partial x_2}(\vec{c}) + \ldots + u_n \frac{\partial f}{\partial x_n}(\vec{c}).$$

Since linear maps can be written as matrixes, $T_c : \mathbb{R}^n \to \mathbb{R}$ is a $[\quad\quad] \leftarrow 1 \times n$ matrix, and multiplication by that matrix acts as above. Thus the matrix representation of T_c is

$$\left[\frac{\partial f}{\partial x_1}(\vec{c}) \quad \frac{\partial f}{\partial x_2}(\vec{c}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\vec{c})\right],$$ <u>when T_c exists</u>!

So our function $f(x,y,z) = \cos(xyz) + xyz + 2z$ from last day has derivative

$$T_c(\vec{u}) = 2u_3 \text{ at } (0,0) \text{ or } [0 \quad 0 \quad 2].$$

Note we are still missing a key ingredient here: Guaranteed existence of $T_c(v)$.

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§12.5

Last day we saw that when $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, there's a way to express the total derivative $T_c(\vec{v})$ in terms of partials $\frac{\partial f}{\partial x_i}(\vec{c})$. This same fact holds in general:

<u>Theorem</u> 12.5: Let $f: S \longrightarrow \mathbb{R}^m$ be differentiable at a point $c \in \text{int}(S) \subset \mathbb{R}^n$. Suppose that $\{u_1, \ldots, u_n\}$ are the unit coordinate vectors and that $\vec{v} = v_1\vec{u}_1 + \ldots + v_n\vec{u}_n$.
Then $T_{\vec{c}}(\vec{v}) = \sum_{k=1}^{n} v_k D_k f(\vec{c})$ (recall $D_k f$ is one of our partial notations).

In particular, the formula we saw last day can be expressed as $T_c(\vec{v}) = \nabla f(\vec{c}) \cdot \vec{v}$ ($\nabla f(\vec{c})$ is gradient) and is a special case of this formula.

<u>Proof</u>: Using linearity:

$$
\begin{aligned}
T_c(\vec{v}) = T_c\left(\sum_{k=1}^{n} v_k \vec{u}_k\right) &= \sum_{k=1}^{n} v_k T_c(\vec{u}_k) \\
&= \sum_{k=1}^{n} v_k f'(\vec{c}; \vec{u}_k) \quad \text{(By Theorem 12.3)} \\
&= \sum_{k=1}^{n} v_k D_k f(\vec{c}) \quad \text{(Since directional derivatives along } u_k \text{ are partials)}
\end{aligned}
$$

<u>Remark</u>: The gradient

$$\nabla f(\vec{c}) = (D_1 f(\vec{c}), D_2 f(\vec{c}), \ldots, D_n f(\vec{c}))$$

becomes important in what follows.

and the defining formula of the total derivative

$$f(\vec{c}+\vec{v}) = f(\vec{c}) + T_c(\vec{v}) + \|\vec{v}\| E_c(\vec{v})$$

becomes

$$f(\vec{c}+\vec{v}) = f(\vec{c}) + \nabla f(\vec{c})\cdot\vec{v} + \underbrace{\|\vec{v}\| E_c(\vec{v})}_{\text{or denote this by } O(\|\vec{v}\|)}, \quad \vec{v}\to 0.$$

We can use this to compute another example of a total derivative T_c (The first one having been computed last day was $f = T_c$ $\forall c$ when f is a linear function).

§12.6: Complex functions (an example of a total derivative).

Suppose that $u, v : \underset{\mathbb{C}}{\overset{\|}{\mathbb{R}^2}} \to \mathbb{R}$ are functions of a complex variable, and define $f: \mathbb{C}\to\mathbb{C}$ (think $\mathbb{R}^2\to\mathbb{R}^2$) by $f = u + iv$. Then if f is differentiable, it must satisfy the Cauchy–Riemann equations

$$D_1 u(\vec{c}) = D_2 v(\vec{c}) \text{ and } D_1 v(\vec{c}) = -D_2 u(\vec{c})$$

at any point $c\in\mathbb{R}^2$ where the derivative exists.

(This is theorem 5.22 in Apostol, which we will not cover here. The point is that functions satisfying the Cauchy-Riemann equations arise naturally, and so the next theorem is indeed interesting.)

Conversely, $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists when the CR eqns are satisfied.

The point is that complex-differentiable functions provide another class of differentiable *real* functions, by viewing $f: \mathbb{C} \longrightarrow \mathbb{C}$ as $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and using the CR equations.

Theorem: Suppose $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, when viewed as a complex function $f(z): \mathbb{C} \longrightarrow \mathbb{C}$ $(z = x+iy)$ is complex-differentiable. Then f is *real* differentiable, and if $f(x,y) = (u(x,y), v(x,y))$ then the total derivative T at $(x,y) \in \mathbb{R}^2$ is the linear function

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Proof: Fix a point (x,y) in the interior of the domain of $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. We need to show that the linear transformation above satisfies

$$f(x+h_1, y+h_2) - f(x,y) = T(h_1,h_2) + \|(h_1,h_2)\| \, E(h_1,h_2)$$

where $E(h_1,h_2) \rightarrow (0,0)$ as $(h_1,h_2) \longrightarrow (0,0)$.

By definition of complex differentiability we have

$$f(z+h) - f(z) = hf'(z) + \|(h_1,h_2)\| \, E(h_1,h_2)$$

Interpreted as an equation relating real numbers, this can be written as

$$f(x+h_1, y+h_2) - f(x,y) = (h_1 + ih_2)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \|(h_1, h_2)\| E(h_1, h_$$

take derivative approaching along x-axis

This term is

$$(h_1 + ih_2)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$= \left(h_1\frac{\partial u}{\partial x} - h_2\frac{\partial v}{\partial x} + i\left(h_2\frac{\partial u}{\partial x} + h_1\frac{\partial v}{\partial x}\right)\right)$$

Thinking of numbers $(a+ib)$ as pairs (a,b), this means (h_1, h_2) is mapped to

$$\left(h_1\frac{\partial u}{\partial x} - h_2\frac{\partial v}{\partial x},\ h_2\frac{\partial u}{\partial x} + h_1\frac{\partial v}{\partial x}\right)$$

The matrix which does this is

$$\begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} \underset{=}{\text{by C.R.}} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Thus we have:

1. Defined "f is differentiable" for $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.
2. Calculated the derivative for functions:
   - $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
   - f linear
   - f complex differentiable.

In general, we need a theorem which says what function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are differentiable, and how to calculate the total derivative in that case.

§12.7 Linear algebra review

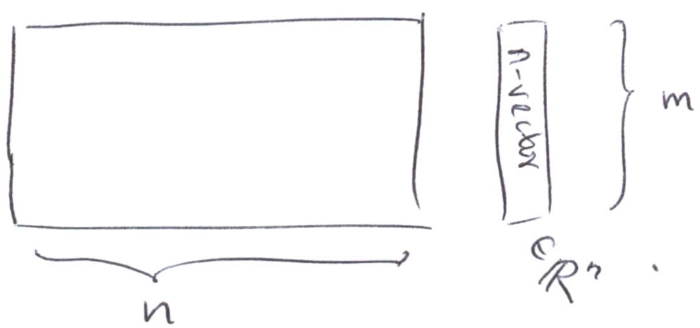
The purpose here is to:

(1) Review linear alg, but also

(2) To establish notation.

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear function. Then we can write T as multiplication by a matrix, which the book denotes by $m(T)$.

Note that the dimensions of $m(T)$ are $m \times n$:



We can convince ourselves that every linear T has a corresponding matrix $m(T)$ by considering the action of T on $\vec{x} \in \mathbb{R}^n$ as follows:

Write

$$\vec{x} = x_1\vec{u}_1 + x_2\vec{u}_2 + \dots + x_n\vec{u}_n$$

where $\vec{u}_i$ is the ith coordinate vector. Then by linearity,

$$T(\vec{x}) = T\left(\sum_{i=1}^{n} x_i\vec{u}_i\right) = \sum_{i=1}^{n} x_i T(\vec{u}_i).$$

However each $T(\vec{u}_i)$ is in $\mathbb{R}^m$ so in terms of unit coordinate vectors in $\mathbb{R}^m$ we can write

$$T(\vec{u}_i) = \sum_{k=1}^{m} t_{i,k}\, \vec{e}_k$$

and then

$$T(\vec{x}) = \sum_{i=1}^{n} x_i T(u_i) = \sum_{i=1}^{n} x_i \sum_{k=1}^{m} t_{i,k} \vec{e}_k = \sum_{k=1}^{m} \Big(\underbrace{\sum_{i=1}^{n} t_{i,k} x_i}_{} \Big) \vec{e}_k$$

dot product

$$\begin{bmatrix} t_{1,k} & t_{2,k} & \cdots & t_{n,k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

So the sum on the right is

$$\begin{bmatrix} t_{1,1} & t_{2,1} & \cdots \\ t_{1,2} & & \\ \vdots & & \\ \vdots & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix},$$ a matrix product.

Knowing $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is represented by a matrix we find the matrix by plugging in u_i, because

$$T(\vec{u}_i) = m(T) \cdot \vec{u} = i^{th} \text{ column of } m(T).$$

Thus, returning to calculus, we know that (by definition) the total derivative of $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at $c \in \mathbb{R}^n$ is a linear function $T_c: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

By Theorem 12.3, $T_c(\vec{v}) = f'(\vec{c}; \vec{v})$ for all $\vec{v} \in \mathbb{R}^n$.

In particular, $T_c(\vec{u}_k) = f'(\vec{c}; u_k) = D_k f(\vec{c})$

$$\phantom{T_c(\vec{u}_k) = f'(\vec{c}; u_k)} \quad \| \quad$$

$\frac{\partial f}{\partial x_k}(\vec{c})$, in other notation.

Recall that we calculate partial derivatives of functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ as follows: If

$f = (f_1, f_2, \ldots, f_m)$ then

$D_k f = (D_k f_1, D_k f_2, D_k f_3, \ldots, D_k f_m)$. Thus to find the k-th column of the matrix T_c, we apply T_c to $\vec{u}_k$ and get

$$T_c(\vec{u}_k) = D_k f(\vec{c}) = (D_k f_1(\vec{c}), D_k f_2(\vec{c}), \ldots, D_k f_m(\vec{c}))^T$$

We use the following notation for the resulting matrix:

$$Df(\vec{c}) = \begin{bmatrix} D_1 f_1(\vec{c}) & D_2 f_1(\vec{c}) & \cdots & D_n f_1(\vec{c}) \\ D_1 f_2(\vec{c}) & D_2 f_2(\vec{c}) & & \vdots \\ \vdots & & & \vdots \\ D_1 f_m(\vec{c}) & \cdots & \cdots & D_n f_m(\vec{c}) \end{bmatrix}$$

and call this matrix the Jacobian matrix of f at $\vec{c} \in \mathbb{R}^n$.

Remark: Despite how we arrived at this matrix, the Jacobian matrix of f is defined at all points $c \in \mathbb{R}^n$ where all partial derivatives $D_k f_i(\vec{c})$ exist.

So, for example, the function

$$f(x,y) = \begin{cases} x+y & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{if otherwise} \end{cases}$$

has a Jacobian at $(0,0)$, it is (here $m=1$)

$$Df(0,0) = \left[\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0) \right] = [1, 1]$$

despite the fact that f is not differentiable at $(0,0)$. I.e the Jacobian exists independent of whether or not f is differentiable at a given point, though if f is differentiable at $\vec{c}$ then certainly $Df(\vec{c})$ gives its total derivative there.