Algebra 3

Lecture 1 Hungerford Section 1.8.

Recall the following:

Definition: Let G, H be groups. Then the set $G \times H$ becomes a group called the direct product when we equip it with the operation

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$. The identity is (e_G, e_H) where e_G, e_H are the identities of G and H respectively, the inverse is: $(g,h)^{-1} = (g^{-1}, h^{-1})$.

We would like to extend this to arbitrary families of groups $\{G_i \mid i \in I\}$, not just products of two groups.

Recall that we can think of elements in the Cartesian product $\prod_{i\in I} G_i$ as functions $f: I \longrightarrow \bigcup_{i\in I} G_i$ satisfying $f(i) \in G_i$ for all i. For example, in the case of a finite product:

$$\left(2,\ (1,3),\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in \mathbb{Z} \times S_3 \times SL_2(\mathbb{R})$$

integers

symmetric group

2×2 matrices with det 1

can be identified with

$$f:\{1,2,3\} \longrightarrow \mathbb{Z} \cup S_3 \cup SL_2(\mathbb{R})$$

given by $f(1)=2$, $f(2)=(1,3)$ and $f(3)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.

Or

$$(1, 2, 3, 4, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \dots$$

can be identified with $f: \mathbb{N} \longrightarrow \mathbb{Z} \cup \mathbb{Z} \cup \dots$ where $f(i)=i$, where i lies in the i^{th} copy of $\mathbb{Z}$.

<u>Definition</u>: Let $\{G_i \mid i \in I\}$ be a family of groups. Define an operation on $\prod_{i\in I} G_i$ as follows: Given $f, g \in \prod_{i \in I} G_i$ (so $f,g : I \longrightarrow \bigcup_{i\in I} G_i$ with $f(i)$ and $g(i)$ in G_i for all i) define $fg : I \longrightarrow \bigcup_{i \in I} G_i$ to be the map $fg(i) = f(i)g(i)$ for all $i \in I$, where multiplication on the right hand side is in G_i. This is called <u>componentwise multiplication</u>, and in the finite case boils down to

$$(g_1, \dots, g_n)(h_1, \dots, h_n) = (g_1h_1, \dots, g_nh_n)$$

for elements $(g_1, \dots, g_n)$ and $(h_1, \dots, h_n)$ of $G_1 \times \dots \times G_n$.

<u>Remark</u>: In place of $\prod_{i \in I} G_i$, when I is finite (say $|I| = n$) it is customary to write

$$G_1 \times \dots \times G_n$$

or $G_1 \oplus \dots \oplus G_n$ if the G_i's are abelian.

<u>Theorem</u>: If $\{G_i \mid i \in I\}$ are groups, then

(i) $\prod_{i \in I} G_i$ is a group, and

(ii) For each $k \in I$, the map $\pi_k : \prod_{i \in I} G_i \longrightarrow G_i$ given by $\pi_k(f) = f(k)$ (or given by $\pi_k(g_1, \ldots, g_n) = g_k$ if I is finite and $|I| = n$) is an onto homomorphism, called a <u>canonical projection</u>.

<u>Proof</u>: A straightforward check.

The direct product of groups can also be described in a more sophisticated way, using what is called "its universal property". Here is the universal property of the direct product stated for the case of two groups G, H and their product $G \times H$:

The group $G \times H$ is the unique group (up to isomorphism) satisfying the following: There are homomorphisms $G \times H \xrightarrow{\pi_G} G$ and $G \times H \xrightarrow{\pi_H} H$ such that for any group K and any homomorphisms $f_G : K \longrightarrow G$, $f_H : K \longrightarrow H$ there exists a unique homomorphism $f : K \longrightarrow G \times H$ such that

$$\pi_G \circ f = f_G$$

and $\pi_H \circ f = f_H$

in other words, we say

$$\begin{tikzcd} K \arrow[r, "f_G"] \arrow[d, "f_H"'] \arrow[dr, dashed, "f"] & G \\ H & G \times H \arrow[l, "\pi_H"] \arrow[u, "\pi_G"'] \end{tikzcd}$$

commutes.

It should be rather obvious that $G \times H$ has this property, because given f_G, f_H as above, the map $f : K \longrightarrow G \times H$ with $f(k) = (f_G(k), f_H(k))$ certainly makes
$$\pi_G \circ f = f_G \quad \text{true.}$$
$$\pi_H \circ f = f_H$$

It is also easy to see that there can be no other map: Suppose $f' : K \longrightarrow G \times H$ were another map with
$$\pi_G \circ f' = f_G$$
$$\pi_H \circ f' = f_H.$$

Then we must have, $\forall k \in K$:
$$\pi_G f'(k) = f_G(k), \text{ so } f'(k) = (f_G(k), ?)$$
$$\pi_H f'(k) = f_H(k), \text{ so } f'(k) = (?, f_H(k)).$$

Overall, $f'(k) = (f_G(k), f_H(k)) = f(k)$

How do we know $G \times H$ is "the unique group" with this property?

Theorem:
Suppose some other group X with homomorphisms $X \xrightarrow{\pi'_G} G$, $X \xrightarrow{\pi'_H} H$ satisfied: For every K with $f_G : K \to G$ and $f_H : K \to H$ there is a unique $f : K \to X$ with $\pi'_G \circ f = f_G$, $\pi'_H \circ f = f_H$. Then X is isomorphic to $G \times H$.

Proof: By the universal property applied to $G \times H$, there exists a unique map $h : X \to G \times H$ such that
(homom)

$$\begin{tikzcd} X \arrow[r, "\pi'_G"] \arrow[d, "\pi'_H"'] \arrow[dr, dashed, "h"] & G \\ H & G \times H \arrow[l, "\pi_H"] \arrow[u, "\pi_G"'] \end{tikzcd}$$

commutes, ie $\pi_G \circ h = \pi'_G$ and $\pi_H \circ h = \pi'_H$.

By the universal property applied to X, there's a unique map $h' : G \times H \to X$ such that

$$\begin{tikzcd} G \times H \arrow[r, "\pi_G"] \arrow[d, "\pi_H"'] \arrow[dr, dashed, "h'"] & G \\ H & X \arrow[l, "\pi'_H"] \arrow[u, "\pi'_G"'] \end{tikzcd}$$

commutes: $\pi'_G \circ h' = \pi_G$
$\pi'_H \circ h' = \pi_H$.

So we can calculate: $\pi'_G \circ h' = \pi_G \Rightarrow \pi_G \circ h \circ h' = \pi_G$

$\pi'_H \circ h' = \pi_H \Rightarrow \pi_H \circ h \circ h' = \pi_H$.

But then we have:

$$\begin{tikzcd} G \times H \arrow[r, "\pi_G"] \arrow[d, "\pi_H"'] \arrow[dr, "h \circ h'"] & G \\ H & G \times H \arrow[l, "\pi_H"] \arrow[u, "\pi_G"'] \end{tikzcd}.$$

By the universal property of $G \times H$, there's a <u>unique</u> map making the diagram above commute, and we know it's the identity $1_{G\times H}: G \times H \longrightarrow G \times H$. So we must have $h \circ h' = id$, and similarly we can show $h' \circ h = id$. Thus $X \cong G \times H$.

Of course this is true for our direct product over a general index set as well:

<u>Theorem</u>: Let $\{G_i \mid i \in I\}$ be a family of groups, and $\{\varphi_i : K \longrightarrow G_i \mid i \in I\}$ a family of homomorphisms. Then there is a unique homomorphism $\varphi : K \longrightarrow \prod_{i \in I} G_i$ such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$.
Moreover, $\prod_{i \in I} G_i$ is the only group which has this property (ie any other group with this property must be isomorphic to $\prod_{i \in I} G_i$).

We can make a slight variation on the direct product:

<u>Definition</u>: The <u>external (weak) direct product</u> of groups $\{G_i \mid i \in I\}$ is denoted $\prod_{i\in I}^{w} G_i$ and is the set of all $f: I \longrightarrow \bigcup_{i\in I} G_i$ with $f(i) \in G_i$ for all i and $f(i) = e_i$ (e_i the identity in G_i) for all but finitely many i. If all G_i are abelian, then $\prod_{i\in I}^{w} G_i$ is typically written $\sum_{i\in I} G_i$ or $\bigoplus_{i\in I} G_i$.

<u>Example</u>: If $\{\mathbb{Z}_i \mid i \in \mathbb{N}\}$ is a collection of copies of $\mathbb{Z}$, then $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_i$ consists of "tuples"

$$(n_1, n_2, n_3, n_4, \ldots\ )$$

where all but finitely many of the n_i's are zero. Addition is still componentwise, as in $\prod_{i\in\mathbb{N}} \mathbb{Z}_i$.

The additional requirement that $f(i)$ is <u>not</u> the identity for all but finitely many i may seem arbitrary and/or artificial.
In fact it behaves much like the product (direct):

<u>Theorem</u>: If $\{G_i \mid i \in I\}$ is a family of groups, then

(i) $\prod_{i\in I}^{w} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$

(ii) For each $k \in I$ the map $i_k : G_k \longrightarrow \prod_{i\in I}^{w} G_i$ given by $i_k(g) = f$, where $f(i) = e_i$ (identity) for $i \neq k$ and $f(k) = g$, is a one-to-one homomorphism.

(iii) For each $k \in I$, $i_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$

<u>Proof</u>: To be done on assignment.

<u>Remarks</u>: ① Note that this is very similar to a theorem we wrote concerning direct products. In fact, it turns out we can also come up with a universal property for this construction.

② When the index set I is finite, direct products $\prod_{i\in I} G_i$ and external weak direct products $\prod_{i\in I}^{w} G_i$ are the same thing.

③ The maps $i_k : G_k \longrightarrow \prod_{i\in I}^{w} G_i$ are called canonical injections.

It is, in fact, fairly natural since the external weak direct product satisfies a universal property (as the direct product did), but we "reverse arrows"

<u>Theorem</u> (Universal property of the weak direct product). Let $\{A_i \mid i \in I\}$ be a family of <u>abelian</u> groups, and B any <u>abelian</u> group. For any family $\{\varphi_i : A_i \to B \mid i \in I\}$ of group homomorphisms there is a unique homomorphism

$$\varphi : \bigoplus_{i \in I} A_i \longrightarrow B$$

such that $\varphi \circ \iota_k = \varphi_k$ for all $k \in I$, and this property determines $\bigoplus_{i \in I} A_i$ up to isomorphism.

<u>Remark</u>: If we omit the word "abelian" the theorem fails. So the external weak direct product is a construction which satisfies a universal property for abelian groups, but <u>not</u> for general groups.

<u>Proof</u>: All groups are abelian, so we write additively.

Let $\{A_i \mid i \in I\}$ be as above, and B some abelian group with homomorphisms $\{\varphi_i : A_i \to B \mid i \in I\}$

To define $\psi : \bigoplus_{i \in I} A_i \longrightarrow B$, let $a \in \bigoplus_{i \in I} A_i$ be an arbitrary element. Then $a(j)$ is nonzero for only finitely many $j \in I$, say it's nonzero for the indices $j_1, j_2, \ldots, j_n$ and that $a(j_k) = a_{j_k} \in A_{j_k}$.

Define $\psi(a) = \psi_{j_1}(a_{j_1}) + \psi_{j_2}(a_{j_2}) + \ldots + \psi_{j_n}(a_{j_n})$.

Since B is abelian and each ψ_j is a homomorphism, it's easy to check that ψ is a homomorphism, too. It also follows from the definition of ψ that

$$\psi \circ i_k = \psi_k \quad \text{for all } k \in I.$$

We only need to check that ψ is unique. So suppose $\varphi : \bigoplus_{i \in I} A_i \longrightarrow B$ were some other map with

$$\varphi \circ i_k = \psi_k \ \ \forall k \in I.$$

Suppose we plug in an arbitrary element $a \in \bigoplus_{i \in I} A_i$ as above. Then $a = i_{j_1}(a_{j_1}) + i_{j_2}(a_{j_2}) + \ldots + i_{j_n}(a_{j_n})$, so

$$\begin{aligned} \varphi(a) &= \varphi\Big(\sum_{k=1}^{n} i_{j_k}(a_{j_k})\Big) \\ &= \sum_{k=1}^{n} \varphi \circ i_{j_k}(a_{j_k}) \\ &= \sum_{k=1}^{n} \psi_{j_k}(a_{j_k}) \\ &= \psi(a), \end{aligned}$$

so that $\varphi = \psi$.

Uniqueness of $\bigoplus_{i \in I} A_i$ follows from this universal property by an argument similar to the case of direct products.

In pictures, recall that if G and H are groups then $G \times H$ was characterized by the diagram:

$$\begin{tikzcd} X \arrow[r, "\pi_G'"] \arrow[d, "\pi_H'"'] \arrow[dr, dashed, "\exists! h"] & G \\ H & G \times H \arrow[l, "\pi_H"] \arrow[u, "\pi_G"'] \end{tikzcd}$$

If we translate the universal property of $A_1 \oplus A_2$ into pictures it becomes:

$$\begin{tikzcd} B & A_2 \arrow[l, "\varphi_2"'] \arrow[d, "i_2"] \\ A_1 \arrow[u, "\varphi_1"] \arrow[r, "i_1"'] & A_1 \oplus A_2 \arrow[ul, dashed, "\exists! \varphi"'] \end{tikzcd}$$

Note that this is "the same picture with arrows reversed".

A natural question to ask at this point is: Given a group G, how do we know if G is secretly a direct product or external weak direct product?

<u>Theorem</u>: Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G, such that

(i) $G = \langle \bigcup_{i \in I} N_i \rangle$ (here, $\langle - \rangle$ means "subgroup generated by")

(ii) For each $k \in I$, $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \langle e \rangle$ ← identity..

Then $G \cong \prod_{i \in I}^{w} N_i$.

<u>Remark</u>: When $|I| < \infty$, since $\prod_{i \in I} G_i = \prod_{i \in I}^{w} G_i$, this also gives a way of recognizing groups as finite direct products.

<u>Sketch of proof</u>: Let $a \in \prod_{i \in I}^{w} N_i$ be an arbitrary element. Then $a : I \longrightarrow \bigcup_{i \in I} N_i$ satisfies $a(i) \in N_i \ \forall i$ and $a(i) = e_i$ (identity) for all ~~by~~ but finitely many $i \in I$. Say that $j_1, j_2, \dots, j_n \in I$ are the elements for which $a(j_k)$ is not the identity,

write $a(j_k) = a_{j_k} \in N_{j_k}$. Then consider the

product $b = a_{j_1} \cdot a_{j_2} \cdot \ldots \cdot a_{j_n} \in G$.

Problem: The element b might not be well-defined, because if we list the a_{j_k}'s in a different order then we get a different expression for b (the a_{j_k}'s are multiplied in a different order).

The following lemma saves us:

Lemma: If N_1 and N_2 are normal in G and $N_1 \cap N_2 = \langle e \rangle$, then $n_1 n_2 = n_2 n_1$ for all $n_2 \in N_2$ and $n_1 \in N_1$.

By this lemma, the element b above ~~is well defined~~ depends only on $a \in \prod_{i \in I}^{w} N_i$, and not on any ordering of $a_{j_1}, a_{j_2}, \ldots, a_{j_n}$. So call b "$\varphi(a)$". This defines a homomorphism $\varphi : \prod_{i \in I}^{w} N_i \longrightarrow G$.

With some work, we can show this is an isomorphism. (Hungerford Theorem 8.6 Ch 1).

Thus we make the following definition:

<u>Definition</u>: Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of G such that $G = \langle \bigcup_{i \in I} N_i \rangle$ and for each $k \in I$, $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \langle e \rangle$. Then G is called the internal weak direct product of $\{N_i \mid i \in I\}$ or the direct sum (internal) if $\{N_i \mid i \in I\}$ are all abelian.