MATH 2080

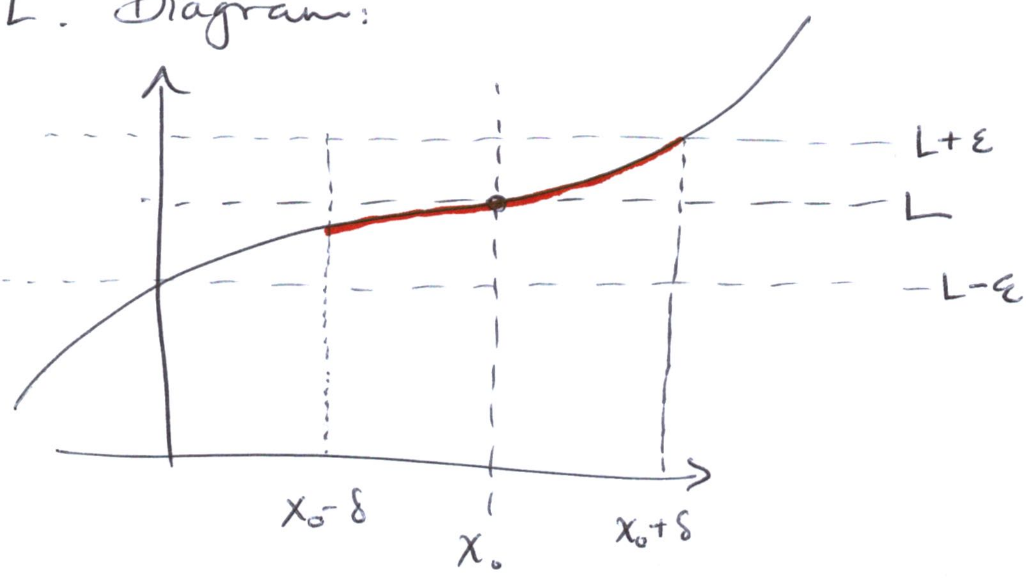
Section 2.1 Limit of a function

Suppose that $D \subseteq \mathbb{R}$ is a set of real numbers and $f : D \to \mathbb{R}$ is a function.

Definition: If x_0 is an accumulation point of D, we write $\lim_{x \to x_0} f(x) = L$ and say "the limit of $f(x)$ as $x \to x_0$ is L" if and only if the following holds: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - x_0| < \delta$ and $x \in D$ implies $|f(x) - L| < \varepsilon$.

Discussion:

The point x_0 must be an accumulation point of the domain D in order for f to be defined near x_0. The value "ε" is how close we want our outputs to be to the limit L, and δ (which often depends on ε) is an instruction as to how close one must be to x_0 to achieve outputs within ε of L. Diagram:



Example: Suppose $f(x) = \frac{x^2-1}{x-1}$, so $D = \mathbb{R} \setminus \{1\}$.

For ~~f~~ $x \neq 1$, $f(x) = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$, so it's a line of slope 1 — except at $x=1$, where there is a gap.

We expect $f(x) \to 2$ as $x \to 1$, since that's how the line $x+1$ behaves. We check:

Let $\varepsilon > 0$. We require $|f(x) - L| < \varepsilon$ for x that are with δ of $x=1$, ie

$$|f(x) - L| = |x+1-2| < \varepsilon$$

$$\iff |x-1| < \varepsilon.$$

So if $\delta = \varepsilon$, then $|x-1| < \delta$ implies $|f(x) - 2| < \varepsilon$.

Example: Define $f: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ by $f(x) = \frac{|x|}{x}$.

For $x < 0$, $f(x) = -1$ and $f(x) = +1$ for $x > 0$.

Consider $\lim\limits_{x \to 0} f(x)$, we will show it does not exist.

Suppose it did, say $\lim\limits_{x \to 0} \frac{|x|}{x} = L$. (for some L).

Now let $\varepsilon > 0$. If $\lim\limits_{x \to 0} \frac{|x|}{x}$ is to exist, then there must be $\delta > 0$ such that being with δ of zero guarantees the outputs are with ε of L.

But the outputs of $f(x)$ are 1 and -1. So if $\varepsilon = \frac{1}{4}$, say, then they cannot be within ε of some number L. Concretely:

Let $\varepsilon = 1/4$. Suppose some $\delta > 0$ gives $0 < |x| < \delta$ $\xrightarrow{\text{implies}}$ $|f(x) - L| < 1/4$. Then $f(-\frac{\delta}{4}) = -1$ and $f(+\frac{\delta}{4}) = +1$, and so we calculate

$$2 = |f(-\tfrac{\delta}{4}) - f(\tfrac{\delta}{4})| \leq |f(-\tfrac{\delta}{4}) - L| + |f(\tfrac{\delta}{4}) - L|$$
$$< \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2},$$

which is clearly a contradiction.

Example: Find $\lim_{x \to 2} f(x)$ where $f(x) = \frac{x-2}{1+x^2}$.

Solution: We know that (from MATH 1500) the limit should be 0. So we try to prove this:

Let $\varepsilon > 0$. We need to find $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|f(x) - 0| < \varepsilon$.

I.e we need to make $\left| \frac{x-2}{1+x^2} - 0 \right| = \frac{|x-2|}{|1+x^2|} < \varepsilon$

by making $|x-2| < \delta$ for some choice of δ. Note that

$\frac{|x-2|}{|1+x^2|} = |x-2| \cdot \frac{1}{|1+x^2|}$, and we already can control the size of $|x-2|$ by simply choosing δ small.

On the other hand, $\frac{1}{|1+x^2|}$ is unbounded if we allow x to be arbitrarily large. So we consider a bound:

If $\delta \le 1$, then $\dfrac{1}{|1+x^2|}$ will be bounded. For if $\delta \le 1$ then $x \in (1,3)$ and so $\dfrac{1}{|1+x^2|}$ is in $\left(\frac{1}{10}, \frac{1}{2}\right)$. Thus if $\delta \le 1$ then $\dfrac{|x-2|}{|1+x^2|} < \dfrac{|x-2|}{2}$,

so choose δ such that

- $\delta \le 1$, and
- $\delta/2 < \varepsilon$, then $\dfrac{|x-2|}{|1+x^2|} < \dfrac{|x-2|}{2} < \dfrac{\delta}{2} < \varepsilon$.

Now we write a nice proof:

Let $\varepsilon > 0$. Choose $\delta = \min\{1, 2\varepsilon\}$. Then if $0 < |x-2| < \delta$

$\delta < 1 \Rightarrow \dfrac{1}{|1+x^2|} < \dfrac{1}{2}$ and

$\delta < 2\varepsilon \Rightarrow |x-2| < 2\varepsilon$ (whenever $0 < |x-2| < \delta$.)

So $\left| \dfrac{x-2}{1+x^2} - 0 \right| = \dfrac{|x-2|}{|1+x^2|} < \dfrac{1}{2} \cdot 2\varepsilon = \varepsilon$.

So $\displaystyle\lim_{x \to 2} \frac{x-2}{1+x^2} = 0$

Example: Define $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{x+3}{1+\sqrt{x}}$.

Show $\displaystyle\lim_{x \to 1} f(x) = 2$.

Solution: Let $\varepsilon > 0$. Find $\delta > 0$ such that $0 < |x-1| < \delta$ implies $|f(x) - 2| < \varepsilon$.

"Solve for" $|x-1|$ as in the last example:

$$|f(x) - 2| < \varepsilon$$

$$\Leftrightarrow \left| \frac{x+3}{1+\sqrt{x}} - 2 \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{x+1-2\sqrt{x}}{1+\sqrt{x}} \right| < \varepsilon \quad \text{(common denominator)}$$

Now make $|x-1|$ appear...

$$\Leftrightarrow \left| \frac{x-1 \;+1\; +1-2\sqrt{x}}{1+\sqrt{x}} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{x-1 + 2(1-\sqrt{x})}{1+\sqrt{x}} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{x-1}{1+\sqrt{x}} + 2\,\frac{1-\sqrt{x}}{1+\sqrt{x}} \right| < \varepsilon$$

$$\frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} = \frac{1-x}{(1+\sqrt{x})^2}$$

$$\Leftrightarrow \left| \frac{x-1}{1+\sqrt{x}} - 2\,\frac{x-1}{(1+\sqrt{x})^2} \right| < \varepsilon$$

$$\Leftrightarrow \left| (x-1)\left(\frac{1}{1+\sqrt{x}} - 2\,\frac{1}{(1+\sqrt{x})^2} \right) \right| < \varepsilon$$

$$\Leftrightarrow \left| (x-1)\left(\frac{\sqrt{x}-1}{(1+\sqrt{x})^2} \right) \right| < \varepsilon$$

$$\Leftrightarrow |x-1| \cdot |\sqrt{x}-1| \cdot \frac{1}{(1+\sqrt{x})^2} < \varepsilon \;.$$

As before: If $0 < |x-1| < \delta$ and $\delta \le 1$, then $x \in (0,2)$ so $0 \le |\sqrt{x} - 1| < 1$. Also

$1 < 1 + \sqrt{x} < 1 + \sqrt{2}$ so $1 < (1+\sqrt{x}) < (1+\sqrt{2})^2$ and

$\frac{1}{(1+\sqrt{2})^2} < \frac{1}{(1+\sqrt{x})^2} < 1$ follows. Thus as long as $\delta \le 1$,

$$|f(x) - 2| = |x-1| \cdot |\sqrt{x} - 1| \cdot \frac{1}{(1+\sqrt{x})^2} < |x-1| \cdot 1 \cdot 1,$$

so if $\delta = \min\{\varepsilon, 1\}$ then

$|f(x) - 2| < |x-1| < \delta = \varepsilon$; so the limit holds.

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Section 2.1 continued. Continuing limit examples.

Example: Define $f:[0,1]\to\mathbb{R}$ by $f(x)=0$ if x is irrational, and if x is rational then define $f(x)=\frac{1}{q}$ where $x=\frac{p}{q}$ (p,q relatively prime).

So $f(0)=0$, $f(\frac{\sqrt{2}}{2})=0$, $f(\frac{3}{4})=\frac{1}{4}$, $f(\frac{13}{15})=\frac{1}{15}$, etc.

Question: Where does f have a limit? Where is there no limit?

Let $x_0\in[0,1]$ be an arbitrary point. Let $\varepsilon>0$ and consider $(x_0-\varepsilon, x_0+\varepsilon)\cap[0,1]$. Since the irrational numbers are dense in $\mathbb{R}$, there are infinitely many irrational numbers in $(x_0-\varepsilon, x_0+\varepsilon)$. So if $\lim\limits_{x\to x_0} f(x)$ exists, it must be equal to zero since $f(y)=0$ for infinitely many $y\in(x_0-\varepsilon, x_0+\varepsilon)$. Is it possible that $\lim\limits_{x\to x_0} f(x)=0$ for some $x_0\in[0,1]$?

This will happen if $f(x)$ is small (ie close to zero) for all x near x_0. This happens if either x is irrational, in which case $f(x)=0$, or if $x=\frac{p}{q}$ with q large, so that $f(x)=\frac{1}{q}$ is small.

Observe that for a fixed q_0, there are only finitely many points $\frac{p}{q}$ with $q\le q_0$, so there are only finitely many points in $[0,1]$ for which $f(x) \geq \frac{1}{q_0}$. We suspect, then, that the limit at <u>any</u> x_0 should be zero — making $f(x)$ as small as we like is only a matter of avoiding finitely many points. So we begin the proof:

Let $\varepsilon > 0$. Choose q_0 such that $\frac{1}{q_0} < \varepsilon$. There are only finitely many points $\frac{p}{q} \in [0,1]$ with $p, q > 0$ and $q \leq q_0$, call them $r_1, r_2, \ldots, r_n$. If it happens $x_0 = r_i$ for some i, then delete that number from the list. Set

$$\delta = \min\{\, |x_0 - r_i| \mid i = 1, \ldots, n \,\},$$

observe $\delta > 0$ since $x_0 \neq r_i$ for all i.

Now if $0 < |x - x_0| < \delta$ and $x \in [0,1]$ then either x is irrational, in which case $f(x) = 0$, or $x = \frac{p}{q}$ with $q > q_0$ so $f(x) = \frac{1}{q} < \frac{1}{q_0} < \varepsilon$. In either case,

$$|f(x) - 0| = |f(x)| \leq \frac{1}{q} < \frac{1}{q_0} < \varepsilon.$$

So in fact, $\lim_{x \to x_0} f(x) = 0$ for all $x_0 \in [0,1]$.

Lesson: It should be clear why our ε-δ notion of limit is needed, if we are to accommodate functions like that in the previous example. No heuristic or hand-waving explanation could have sufficed.

§ 2.2 Limits of functions and sequences.

We investigate the relationship between sequences converging to x_0 and functions f that have a limit as $x \to x_0$.

Consider $f: D \to \mathbb{R}$, x_0 an accumulation point of D and $\lim_{x \to x_0} f(x) = L$. Then if $x_n \in D$ for all n and $\{x_n\}_{n=1}^{\infty}$ converges to x_0, then $\{f(x_n)\}_{n=1}^{\infty}$ must approach L as x_n approaches x_0, indeed this turns out to be true. In fact there is a converse..

__Theorem__: If $f: D \to R$ with x_0 an accumulation point of D then $\lim_{x \to x_0} f(x)$ exists if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D$ for all x_n (and $x_n \neq x_0$) converging to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges.

__Think__: Continuous functions send convergent sequences to convergent sequences..

__Remark before proof:__

Note that if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both satisfy the hypotheses of the theorem, then $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ both converge.

We suspect this theorem implies they converge to the same limit, and indeed it does: If $z_{2n} = x_n$ and $z_{2n+1} = y_n$ then $\{z_n\}_{n=1}^{\infty}$ is a sequence converging to x_0 and thus $\{f(z_n)\}_{n=1}^{\infty}$ converges. Both $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are subsequences of $\{f(z_n)\}_{n=1}^{\infty}$ and so must converge to the limit of $\{f(z_n)\}_{n=1}^{\infty}$.

In the proof we will see that actually $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{x\to x_0} f(x)$ for every sequence $\{x_n\}_{n=1}^{\infty}$ in the domain with $x_n \to x_0$.

<u>Proof</u>: Suppose $\lim\limits_{x\to x_0} f(x) = L$. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence converging to x_0, $x_n \in D$ and $x_n \neq x_0$ for all n. Consider $\{f(x_n)\}_{n=1}^{\infty}$, and let $\varepsilon > 0$. By ~~convergence~~ continuity of $f(x)$ there's $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$. But by convergence of $\{x_n\}_{n=1}^{\infty}$ to x_0 there's N such that $n \geq N$ implies $|x_n - x_0| < \delta$. Thus for $n \geq N$ $|f(x_n) - L| < \varepsilon$, so $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Suppose on the other hand that all sequences $\{x_n\}_{n=1}^{\infty}$ converging to x_0 give rise to convergent sequences $\{f(x_n)\}_{n=1}^{\infty}$, which (by our remarks) must all have a common limit L.

Suppose $\lim_{x \to x_0} f(x) \neq L$. So, $\exists\, \varepsilon > 0$ such that for all $\delta > 0$ there is $x \in D$ with $0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon$. In particular, for each $\delta = \frac{1}{n}$ there's x_n with $0 < |x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon$. But now $\{x_n\}_{n=1}^{\infty}$ converges to x_0 and is a sequence of numbers in the domain all distinct from x_0, so $\{f(x_n)\}_{n=1}^{\infty}$ should converge to L. This contradicts $|f(x_n) - L| \geq \varepsilon$ for all n. So it must be that $\lim_{x \to x_0} f(x) = L$.

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§2.2 continued

Similar to last day, we can also prove:

<u>Theorem</u>: Let $f: D \to \mathbb{R}$ and suppose x_0 is an accumulation point of D. If, for every sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D \setminus \{x_0\}$ for each n, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy, then f has a limit at x_0.

<u>Proof</u>: This follows from our last theorem, and the fact that convergent $\Longleftrightarrow$ Cauchy.

Similarly we can prove:

<u>Theorem</u>: Let $f: D \to \mathbb{R}$ and suppose x_0 is an accumulation point of D. If f has a limit at x_0, then there's a neighbourhood Q of x_0 and $M \in \mathbb{R}$ such that for all $x \in Q \cap D$, $|f(x)| \leq M$.

<u>Proof</u>: We could prove this using the same theorem that we used to prove the last theorem, or prove it directly. A direct proof goes like this:

Set $L = \lim_{x \to x_0} f(x)$, and let $\varepsilon = 1$. Since f has a limit at x_0, for $\varepsilon = 1$ there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon = 1$.

Now x_0 is an accumulation point of D. If it's actually <u>in</u> D, then set

$M = \max\{|L-1|, |L+1|, |f(x_0)|\}$

and if it's <u>not</u> in D then set

$M = \max\{|L-1|, |L+1|\}$, set $Q = (x_0 - \delta, x_0 + \delta)$.

<u>Then</u> it follows that if $x \in Q \cap D$, then $|f(x)| \leq M$.

Informally we could say: If $f(x)$ has a limit ~~at~~ at x_0, then f is bounded 'near' x_0. Ie there's a nbhd of x_0 where $f(x)$ is bounded.

So E.g. $\frac{1}{x}$ at $x=0$ has no limit since $\frac{1}{x}$ is unbounded in every nbhd of 0.

<u>Example</u>: Consider the function $f:(0,1) \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{p} + \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q},\ p, q \text{ relatively prime} \end{cases}$$

Does $\lim_{x \to 0} f(x)$ exist?

In light of the previous theorems, it suffices to produce two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converging to 0 such that $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ converge to different limits.

Set $x_n = \frac{1}{n}$. Then $\{x_n\}_{n=1}^{\infty}$ converges to 0, and $f(x_n) = 1 + \frac{1}{n}$, so $\{f(x_n)\}_{n=1}^{\infty}$ converges to 1.

On the other hand, note that for all n, n and $n+1$ are relatively prime. (If d divides n and $n+1$, then d divides $n+1-n=1$). So choose $y_n = \frac{n}{(n+1)^2}$. Then $y_n = \frac{n}{n^2+2n+1}$ converges to 0 (that is, $y_n = \frac{1/n}{1+2/n+1/n^2}$ so $n \to \infty$ gives $\frac{0}{1}=0$).

and $f(y_n) = \frac{1}{n} + \frac{1}{(n+1)^2}$, and so $\{f(y_n)\}_{n=1}^{\infty}$ also converges to zero. Thus the function f has no limit at $x=0$, since we've found sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ contradicting our theorem.

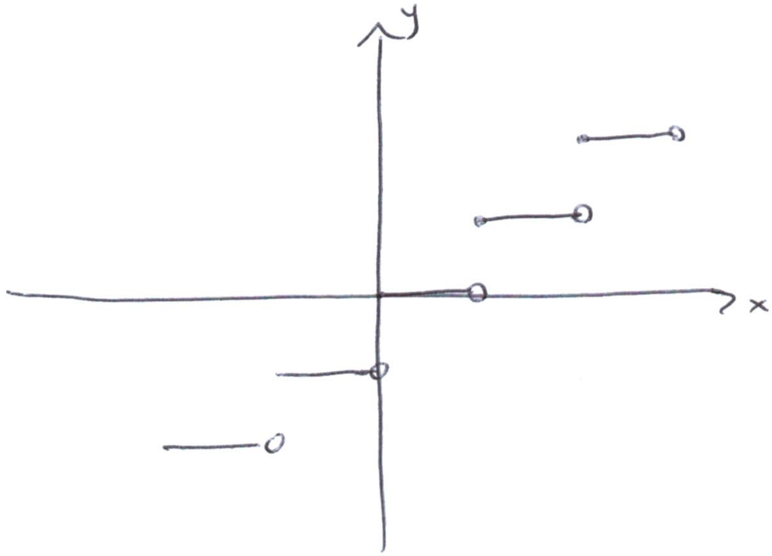
<u>Example</u>: Given $x \in \mathbb{R}$, let

$f(x) = [x] =$ largest integer less than or equal to x

$[n] = n$ if $n \in \mathbb{Z}$

$[\pi] = 3$

$[\frac{4}{3}] = 1$, etc. The sketch is:



Thus we expect $\lim_{x \to n} [x]$ does not exist when $n \in \mathbb{Z}$.

When x_0 is <u>not</u> an integer, set δ = distance from x_0 to nearest integer. Then if $0 < |x - x_0| < \delta$, $[x] = [x_0]$ so $|f(x) - [x_0]| = 0 < \varepsilon$ for all $\varepsilon > 0$.

So $f(x) = [x]$ has a limit at all $x_0 \notin \mathbb{Z}$. On the other hand, for $x_0 \in \mathbb{Z}$ consider $\left\{x_0 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

If n odd, then $(-1)^n = -1$ and so

$$x_0 + \frac{(-1)^n}{n} = x_0 - \frac{1}{n} < x_0, \text{ so}$$

$$f(x_0 - \tfrac{1}{n}) = [x_0 - \tfrac{1}{n}] = [x_0] - 1$$

and if n is even, then $(-1)^n = 1$ and so

$$x_0 + \frac{(-1)^n}{n} = x_0 + \frac{1}{n} > x_0, \text{ therefore}$$

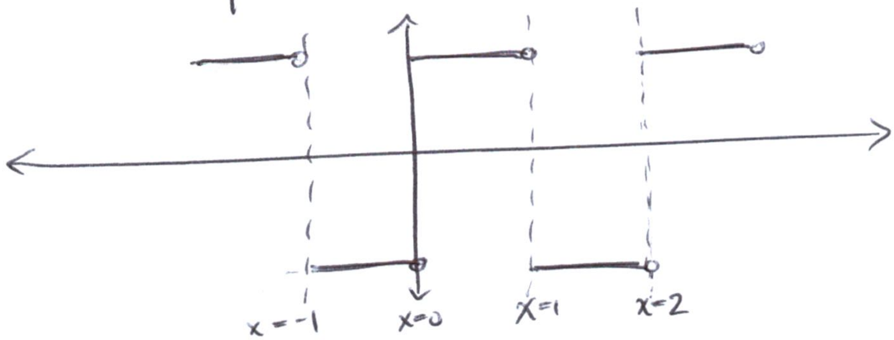
$$f\left(x_0 + \frac{1}{n}\right) = \left[x_0 + \tfrac{1}{n}\right] = [x_0].$$

Then, since $\left\{x_0 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ converges to x_0, yet $\left\{f\left(x_0 + \frac{(-1)^n}{n}\right)\right\}_{n=1}^{\infty}$ does not converge since the subsequences $\left\{f\left(x_0 + \frac{(-1)^{2n}}{2n}\right)\right\}_{n=1}^{\infty}$ and $\left\{f\left(x_0 + \frac{(-1)^{2n+1}}{2n+1}\right)\right\}_{n=1}^{\infty}$ converererge to different limits, so we conclude f does not have a limit at $x_0 \in \mathbb{Z}$.

Example: Define

$$f(x) = \begin{cases} -1 & \text{if } [x] \text{ is odd} \\ +1 & \text{if } [x] \text{ is even} \end{cases}$$

So $f(x)$ is a square wave.



Consider the function $f\left(\frac{1}{x}\right)$. Does $\lim_{x\to 0} f\left(\frac{1}{x}\right)$ exist?

Consider the sequences $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{2n+1}$.

Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converge to 0. Yet if we plug them into the function $f\left(\frac{1}{x}\right)$, we get:

$f\left(\frac{1}{x_n}\right) = f\left(\frac{1}{1/2n}\right) = f(2n) = +1$, so $\left\{f\left(\frac{1}{x_n}\right)\right\}_{n=1}^{\infty}$ converges to 1 (it's constant, actually). We also compute

$f\left(\frac{1}{y_n}\right) = f\left(\frac{1}{1/2n+1}\right) = f(2n+1) = -1$, so $\left\{f\left(\frac{1}{y_n}\right)\right\}_{n=1}^{\infty}$ converges to -1. Thus $\lim_{x\to 0} f\left(\frac{1}{x}\right)$ does not exist.