MATH 2080

Section 1.4 continued

Theorem: A sequence converges if and only if all of its subsequences converge in which case they all converge to the same limit.

Proof: First, if every subsequence converges then it's easy to conclude that the sequence itself converges - because every sequence is a subsequence of itself.

Conversely, suppose $\{a_n\}_{n=1}^{\infty}$ converges to A and that $\{a_{n_k}\}_{k=1}^{\infty}$ is any subsequence. Let $\varepsilon > 0$ be given. Then choose N such that $|a_n - A| < \varepsilon$ for all $n \geq N$. Now since $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$, we have $n_1 < n_2 < n_3 < \dots$, in particular n_k is always at least as large as k. So for $k \geq N$, $n_k \geq N$, hence $|a_{n_k} - A| < \varepsilon$ for $k \geq N$. Thus $\{a_{n_k}\}_{k=1}^{\infty}$ converges to A.

We can slightly weaken the requirement that <u>all</u> subsequences converge if we assume the sequence is bounded, and arrive at

Theorem: Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. If all of its convergent subsequences converge to the same limit, then $\{x_n\}_{n=1}^{\infty}$ converges.

Proof: Assume that all convergent subsequences of $\{x_n\}_{n=1}^{\infty}$ have the same limit - call it x_0. (By Exercise 36 there's at least one convergent subsequence of $\{x_n\}_{n=1}^{\infty}$).

Now suppose $\{x_n\}_{n=1}^{\infty}$ does not converge to x_0, we will find a contradiction. Under this assumption, there exists $\varepsilon > 0$ for which we cannot find N such that $n \geq N$ implies $|x_n - x_0| < \varepsilon$. So, if we tried $N = 1$, there would be an x_{n_1} with $n_1 > 1$ and $|x_{n_1} - x_0| \geq \varepsilon$. If we tried $N = 2$ it would also fail, meaning we could find $n_2 > 2$ (and also $n_2 \neq n_1$) such that $|x_{n_2} - x_0| \geq \varepsilon$ for some x_{n_2}. Continuing, if we tried $N = k$ for any k, we would always $n_k > k$ (and not equal to any of $n_1, n_2, \ldots, n_{k-1}$) such that $|x_{n_k} - x_0| \geq \varepsilon$.

But now $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, and since $\{x_n\}_{n=1}^{\infty}$ is bounded, so is $\{x_{n_k}\}_{k=1}^{\infty}$. Being a bounded sequence, $\{x_{n_k}\}_{k=1}^{\infty}$ will have a convergent subsequence (exercise!), which by our choices cannot converge to x_0.

This contradicts our assumption that every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x_0.

These last two theorems may seem like bizarre special cases, but the are in fact very useful in studying some real-life sequences that arise naturally.

For example, we now know that it suffices to find one divergent subsequence of $\{a_n\}_{n=1}^{\infty}$ to prove that $\{a_n\}_{n=1}^{\infty}$ diverges.

<u>Example</u>: Consider the sequence $\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right\}_{n=1}^{\infty}$, ie the terms are 1, $1+\frac{1}{2}$, $1+\frac{1}{2}+\frac{1}{3}$, and in general set

$$a_n = \sum_{k=1}^{n} \frac{1}{k}.$$

We saw already that this sequence diverges, by some fairly tricky arithmetic. Consider the following alternative proof:

- The term a_9 is a sum of reciprocals of one-digit numbers, there are nine such terms and each is bigger than $1/10$. So $a_9 > 9\cdot\left(\frac{1}{10}\right) = \frac{9}{10}$.
- The term a_{99} is a sum of reciprocals of nine one digit numbers (each bigger than $1/10$) and 90 reciprocals of 2-digit numbers, each bigger than $\frac{1}{100}$. So $a_{99} > \frac{9}{10} + 90\cdot\left(\frac{1}{100}\right) = 2\left(\frac{9}{10}\right)$

In general, continuing this pattern gives $a_{10^k-1} > k\left(\frac{9}{10}\right)$. So the subsequence $\{a_{10^k-1}\}_{k=1}^{\infty}$ is unbounded, thus diverges. Therefore so does $\{a_n\}_{n=1}^{\infty}$.

Example: The rational numbers in the interval $[0,1]$ form a countable set, so we can list the elements of $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, \ldots\}$. Thus they form a sequence $\{a_n\}_{n=1}^{\infty}$. No matter how we have enumerated them, this sequence cannot converge because it contains many subsequences that converge to different limits (despite the sequence itself being bounded). For example, $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ no matter the enumeration we've chosen, as is $\{1-\frac{1}{n}\}_{n=1}^{\infty}$. Yet one converges to 0, the other to 1. Thus $\{a_n\}_{n=1}^{\infty}$ cannot converge, by our second theorem.

Example: Consider $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}$. The subsequences

$$\left\{\frac{1+(-1)^{2n}}{2}\right\}_{n=1}^{\infty} = \{1, 1, 1, \ldots\} \text{ and}$$

$$\left\{\frac{1+(-1)^{2n+1}}{2}\right\}_{n=1}^{\infty} = \{0, 0, 0, \ldots\} \text{ converge to } 0$$

and 1 respectively.

and thus $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}$ cannot converge.

Last, we want to consider sequences whose terms either become larger or smaller with every step, because they have very special behaviours.

<u>Definition</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ is <u>increasing</u> if $a_n \leq a_{n+1}$ for all positive integers n. A sequence is <u>decreasing</u> if $a_n \leq a_{n+1}$ for all positive integers n. A sequence is <u>monotone</u> iff it is either increasing or decreasing.

<u>Example</u>: The sequence $\left\{\frac{1+(-1)^{2n}}{2}\right\}_{n=1}^{\infty} = \{1, 1, 1, \ldots\}$ is increasing, decreasing, and monotone.

Easy fact to prove: A sequence is both increasing <u>and</u> decreasing if and only if it is constant, i.e. $a_n = a_{n+1}$ for all n.

<u>Theorem</u>: A monotone sequence is convergent if and only if it is bounded.

<u>Proof</u>: Suppose $\{a_n\}_{n=1}^{\infty}$ is monotone, say it's increasing (the case of decreasing being similar). If we assume $\{a_n\}_{n=1}^{\infty}$ is bounded, then we can say $s = \sup\{a_n\}_{n=1}^{\infty}$ exists. We'll show $\{a_n\}_{n=1}^{\infty}$

converges to s.

To see this, let $\varepsilon > 0$. Then there exists n_0 such that $s - \varepsilon < a_{n_0}$. Then for $n \geq n_0$,

$$s - \varepsilon < a_{n_0} \leq a_n \leq s < s + \varepsilon,$$

hence $n \geq n_0$ implies $|a_n - s| < \varepsilon$. Thus $\{a_n\}_{n=1}^{\infty}$ converges to s.

Conversely, we already know that if a sequence converges then it is bounded.

MATH 2080

Section 1.4 continued.

A few examples to clarify what we've just learned.

Example: Define a sequence $\{s_n\}_{n=1}^{\infty}$ as follows

$$s_1 = \sqrt{2}, \quad \text{and} \quad s_n = \sqrt{2+\sqrt{s_{n-1}}} \quad \text{for } n = 2, 3, 4 \ldots$$

We'll show to analyze this sequence using the tools we have at hand.

Claim: $s_n \leq 2$ for all n, and $s_{n+1} \geq s_n$ for all n, in other words, the sequence is increasing and bounded above by 2. We'll show this by induction.

First, note that $s_2 = \sqrt{2+\sqrt[4]{2}} \geq \sqrt{2} = s_1$, so the claim $s_{n+1} \geq s_n$ holds for $n=1$. Suppose it is true for $n=k$, so $s_{k+1} \geq s_k$. Then

$$\sqrt{s_{k+1}} \geq \sqrt{s_k}$$

therefore
$$s_{k+2} = \sqrt{2+\sqrt{s_{k+1}}} \geq \sqrt{2+\sqrt{s_k}} = s_{k+1};$$

so the sequence is increasing.

Also if we assume $s_k \leq 2$ then

$$s_{k+1} = \sqrt{2+\sqrt{s_k}} \leq \sqrt{2+\sqrt{2}} \leq \sqrt{2+2} = 2.$$

So it's an increasing sequence bounded above by 2, and thus has a limit L.

THere is a trick for determining the value of L, we do not cover it here (~~we haven't~~ prepared the necessary background material yet.)

The missing ingredient is a proof that if $\{a_n\}_{n=1}^{\infty}$ converges to A, then $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{A}$, a proof of which appears in the next chapter — so we will return to this then.

<u>Theorem</u>: Let E be any set of real numbers. Then x_0 is an accumulation point of E iff there is a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E such that $\{x_n\}_{n=1}^{\infty}$ converges to x_0, with all x_n distinct from x_0.

<u>Proof</u>: First, suppose x_0 is an accumulation point of E and we'll construct a sequence $\{x_n\}_{n=1}^{\infty}$ using elements of E that converges to x_0:

Because x_0 is an accumulation point, for each $\varepsilon = \frac{1}{n}$ there is a point x_n, which is not equal to x_0, contained in $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$. Then, given $\varepsilon > 0$, if we choose $N > \frac{1}{\varepsilon}$ then for all $n \geq N$ we have $|x_n - x_0| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. So in fact, $\{x_n\}_{n=1}^{\infty}$ converges to x_0.

Conversely suppose $\{x_n\}_{n=1}^{\infty}$ converges to x_0, and that x_k are all elements of E and $x_k \neq x_0 \ \forall k$. Then every neighbourhood of x_0 contains all but finitely many terms in the sequence $\{x_n\}_{n=1}^{\infty}$, so every neighbourhood of x_0

contains infinitely many points of E. Thus x_0 is an acc. point of E.

Let's do a few more examples to study certain tricks.

<u>Example</u>: Suppose $0 < b < 1$ and consider $\{b^n\}_{n=1}^{\infty}$.
We know from "calculus-type" (math 1500) reasoning that the sequence converges to 0, but to prove it carefully we might do:

First observe that $b^{n-1} - b^n = b^{n-1}(1-b) > 0$, so that $b^{n-1} > b^n$ and the sequence is decreasing. Moreover the sequence is bounded below (by 0) so it must converge, call its limit L. The subsequence $\{b^{2n}\}_{n=1}^{\infty}$ must also converge to L, but if we consider b^{2n} as the product $b^n \cdot b^n$ then the sequence $\{b^n \cdot b^n\}_{n=1}^{\infty}$ must converge to the product of the limits of $\{b^n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, ie L^2. By uniqueness of the limit, this forces $L^2 = L$ or $L = 0, 1$. We know $L = 1$ is impossible since $b < 1$ and the sequence is decreasing. Thus the limit is 0.

In order to illustrate another trick, let us use the fact that $\{a_n\}_{n=1}^{\infty}$ converging to A

means $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{A}$. (Proof in exercises).

<u>Example</u>: Suppose $0 < c < 1$ and consider $\{\sqrt[n]{c}\}_{n=1}^{\infty}$.

We calculate

$$c^{1/n} - c^{1/n+1} = c^{1/n}\left(1 - c^{\frac{1}{n+1} - \frac{1}{n}}\right)$$

$$= c^{1/n}\left(1 - c^{\frac{-1}{(n+1)n}}\right) < 0 \quad \text{since } c < 1$$

So $c^{1/n} < c^{1/n+1}$ and the sequence is increasing, it's bounded above by 1. So the sequence converges to some number L.

Now the sequence $\{\sqrt{\sqrt[n]{c}}\}_{n=1}^{\infty}$ must converge to $\sqrt{L}$, but note that $\sqrt{\sqrt[n]{c}} = \sqrt[2n]{c}$, so $\{\sqrt{\sqrt[n]{c}}\}_{n=1}^{\infty}$ is the same as the subsequence $\{\sqrt[2n]{c}\}_{n=1}^{\infty}$, which (as it is a subsequence of the original) must converge to L. By uniqueness of limits, $\sqrt{L} = L$ and so $L = 0$ or 1, L cannot be 0 since $c > 0$ and it's an increasing sequence. Thus $L = 1$.

<u>Remark</u>: The text contains several more examples not covered here.

MATH 2080.

Uncountability of the real numbers.

Recall that a set S is uncountable if there is no bijective function $f: \mathbb{N} \longrightarrow S$. Today we show that $\mathbb{R}$ is uncountable, using what we've learned about sequences. This lecture is a combination of projects 0.1 and 1.4.

First, note that there are many proofs of the fact that $\mathbb{R}$ is uncountable, the most famous is "Cantor's diagonal argument". It appears in many popular math videos, but they're almost all wrong. (The problem is that they use decimal representations in Cantor's argument, and almost always overlook the fact that ~~every~~ real numbers ~~has mor~~ can have more than one decimal representation, e.g. $0.999\ldots = 1$.)

Here is our proof that $\mathbb{R}$ is uncountable.

Project 0.1

Show that $(0,1)$ is equivalent to $[0,1]$.

Proof: Define $f: (0,1) \longrightarrow \mathbb{R}$ as follows:

$$f\left(\frac{1}{n}\right) = \frac{1}{n-1} \quad \text{for all numbers in } (0,1) \text{ of the form } \frac{1}{n} \text{ where } n \in \mathbb{N} \text{ and } n \geq 2$$

and $f(x) = x$ for all other $x \in (0,1)$.

step ① First we show that f is a 1-1 function from $(0,1)$ to $(0,1]$. To see this, first observe that if $x \in (0,1)$ then $f(x) = x$ gives $f(x) \in (0,1)$, while if $\frac{1}{n} \in (0,1)$ then $f(\frac{1}{n}) = \frac{1}{n-1}$ gives $f(x) \in (0,1]$, since $f(\frac{1}{2}) = 1$. So the image of f is indeed in $(0,1]$.

To see f is 1-1, suppose $f(x) = f(y)$. If x is of the form $\frac{1}{n}$ and y is not of this form, then $f(x) = f(y)$ cannot happen. So either

① • x and y are both $\frac{1}{n}$ and $\frac{1}{m}$ for some n, m,

or ② • x and y are both not of the form $\frac{1}{n}$.

In case ①, $f(\frac{1}{m}) = f(\frac{1}{n}) \Rightarrow \frac{1}{m-1} = \frac{1}{n-1} \Rightarrow n = m$,

and in case ② $f(x) = f(y) \Rightarrow x = y$, so f is 1-1.

step ② The function f is onto $(0,1]$, because if $y \in (0,1]$ is not of the form $\frac{1}{n}$ then $f(y) = y$, while if it is of the form $\frac{1}{n}$ then $f(\frac{1}{n+1}) = \frac{1}{n}$.

step ③ Now define a function $g : [0,1) \longrightarrow [0,1]$ by

$g(x) = f(x)$ for all $x > 0$

and $g(0) = 0$. Then $g(x)$ is 1-1 and onto, since $f(x)$ is 1-1 and onto.

step 4 Last in our collection of functions, we can define $h : [0,1) \longrightarrow (0,1]$ by $h(x) = x$ for all $x \in (0,1)$, and $h(0) = 1$. Then h is 1-1 and onto, so $[0,1)$ and $(0,1]$ are equivalent.

step 5 Put together all the functions we've built:

$$(0,1) \xrightarrow{f} (0,1] \xrightarrow{h^{-1}} [0,1) \xrightarrow{g} [0,1].$$

Each of f, h^{-1} and g is 1-1 and onto, so the function

$g \circ h^{-1} \circ f : (0,1) \longrightarrow [0,1]$ is 1-1 and onto. Thus $(0,1)$ is equivalent to $[0,1]$.

——— Project 0.1 done ———

Project 1.4

Now we'll prove that $\mathbb{R}$ is uncountable, by showing that $[0,1]$ is uncountable. Project 1.4 suggests, as a first step, showing that $\mathbb{R}$ is equivalent to $[0,1]$ — but this is unnecessary instead we observe:

Lemma (in place of the suggested step 1):

If $[0,1]$ is uncountable, then so is $\mathbb{R}$.

Proof: Equivalently we can show that if $\mathbb{R}$ is countable, then so is $[0,1]$. But this is a consequence of our theorem that states "subsets of countable sets are countable".

Thus it remains to show that $[0,1]$ is uncountable, for which we follow steps ② – ⑤ of project 1.4.

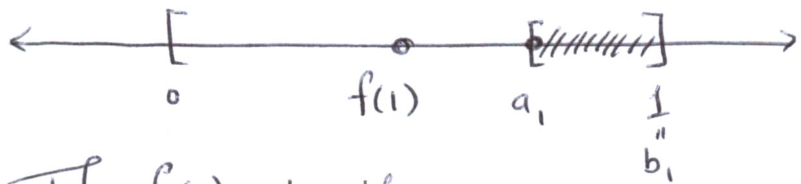
<u>Theorem</u> $[0,1]$ is uncountable.

<u>Proof</u>: Suppose not, say it's countable and $f: \mathbb{N} \to [0,1]$ is 1-1 and onto. We will show this is impossible.

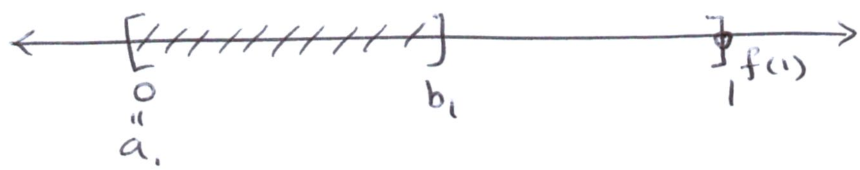
<u>Step ①</u> First we define sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $[a_1, b_1] \subset [0,1]$ and for each $n \in \mathbb{N}$ $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and such that $f(n) \notin [a_n, b_n]$.

We define such sets recursively. To define $[a_1, b_1]$ we consider two cases:

<u>Case 1</u>: If $f(1) \in [0,1)$ then set $a_1 = \frac{f(1)+1}{2}$, and $b_1 = 1$.



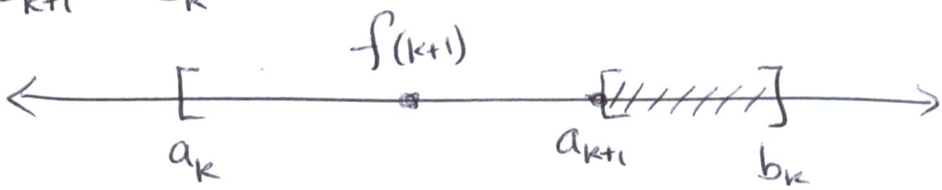
<u>Case 2</u>: If $f(1) = 1$ then set $a_1 = 0$ and $b_1 = 1/2$



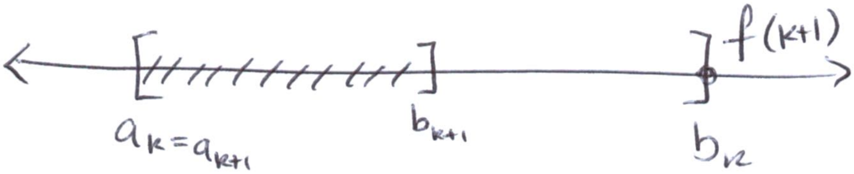
In general, suppose you've defined $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ so that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for $n = k-1, \ldots, 0$ and $f(n) \notin [a_n, b_n]$. Define a_{k+1} and b_{k+1} according to the ~~two~~ three cases!

Case 1: If $f(k+1)$ is not in $[a_k, b_k]$ then set $a_{k+1} = a_k$, $b_{k+1} = b_k$ (ie do nothing).

Case 2: If $f(k+1) \in [a_k, b_k)$ then set $a_{k+1} = \frac{f(k+1) + b_k}{2}$ and $b_{k+1} = b_k$



Case 3: If $f(k+1) = b_k$ then set $a_{k+1} = a_k$ and $b_{k+1} = \frac{b_k + a_k}{2}$



By construction, our sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the necessary properties.

Step 3 The sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n. It's bounded above by 1 since $[a_1, b_1] \subset [0,1]$ and $a_n \le b_1$ for all n. So it converges, call the limit A.

Step 4 Next we observe that $A \in [a_n, b_n]$ for all n. To see this, suppose the opposite — that there is some $n > 0$ such that $A \notin [a_n, b_n]$.

~~Then $[a_n, b_n]$ is a neighbourhood of~~

Suppose $A > b_n$, and set $\varepsilon = A - b_n$. Then the neighbourhood $(A-\varepsilon, A+\varepsilon)$ may contain some of the points $\{a_1, a_2, \ldots, a_{n-1}\}$ but it contains none of $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ since they're all less than $b_n = A - \varepsilon$.

If $A < a_n$ then if $\varepsilon = a_n - A$ the neighbourhood $(A-\varepsilon, A+\varepsilon)$ similarly can contain at most finitely many a_n.

In either case, the conclusion contradicts $\{a_n\}_{n=1}^{\infty}$ converging to A. Thus $A \in [a_n, b_n]$ for all n.

Thus, given $m \in \mathbb{N}$ we know $A \neq f(m)$ since $A \in [a_m, b_m]$ while $f(m) \notin [a_m, b_m]$. So $A \notin \mathrm{im}(f)$.

step 3 We conclude that $[0,1]$ must be uncountable, since our arbitrary function $f: \mathbb{N} \to [0,1]$ turned out to be not onto, since A is not in its image.

So, $[0,1]$ is uncountable and thus so is $\mathbb{R}$.