MATH 2080

Section 1.3 Arithmetic operations on sequences.

We now learn how to effectively take limits via arithmetic operations on sequences.

For example, suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences. Then what happens with $\{a_n+b_n\}_{n=1}^{\infty}$, $\{a_n b_n\}_{n=1}^{\infty}$ and $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ (if b_n is never zero).

__Theorem__: If $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B, then $\{a_n+b_n\}_{n=1}^{\infty}$ converges to $A+B$.

__Proof__: Choose $\varepsilon>0$ arbitrary. Then there's an integer N_1 such that $n \geq N_1$ implies $|a_n - A| < \varepsilon/2$, and there's an integer N_2 such that $n \geq N_2$ implies $|b_n - B| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$.

Then if $n \geq N$, we have __both__ $|a_n - A| < \varepsilon/2$ and $|b_n - B| < \varepsilon/2$, so this lets us compute:

$$
\begin{aligned}
|(a_n+b_n)-(A+B)| &= |a_n - A + b_n - B| \\
&\leq |a_n - A| + |b_n - B| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

So overall, $\{a_n+b_n\}_{n=1}^{\infty}$ converges to $A+B$.

We can do the same for $\{a_n b_n\}_{n=1}^{\infty}$. First, a lesson in epsilon-picking: Here is how to write a nice proof that involves choosing big values of N.

Suppose $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B. We guess that $\{a_n b_n\}_{n=1}^{\infty}$ converges to AB. So we want

$$|a_n b_n - AB| < \varepsilon$$

by choosing n to be large. Can we bound $|a_n b_n - AB|$ above by the quantities $|a_n - A|$ and $|b_n - B|$ somehow?

$$\begin{aligned}
|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\
&\leq |a_n b_n - a_n B| + |a_n B - AB| \\
&= |a_n| \underbrace{|b_n - B|}_{\text{here is the quantity we wanted to see.}} + |a_n B - AB| \\
&= |a_n||b_n - B| + |B| \underbrace{|a_n - A|}_{\text{here is the other quantity we wanted to see.}}
\end{aligned}$$

Now by choosing n large, we can make both $|a_n - A|$ and $|b_n - B|$ small. The constant $|B|$ is negligible if we choose $|a_n - A|$ small enough, but $|a_n|$ is not negligible — it depends on n. However, convergent sequences are bounded, so there exists M such that

$|a_n| \le M$ for all n. (So $|a_n|$ is never too large).

Now given $\varepsilon > 0$, we want to get

$$|a_n b_n - AB| \le |a_n||b_n - B| + |B||a_n - A| < \varepsilon$$

This will happen as long as

$$|b_n - B| < \frac{\varepsilon}{2|M|} \quad \text{and} \quad |a_n - A| < \frac{\varepsilon}{2|B|}.$$

So we could choose n so large that these both hold, and get

$$\begin{aligned} |a_n b_n - AB| &\le |a_n||b_n - B| + |B||a_n - A| \\ &\le |M||b_n - B| + |B||a_n - A| \\ &< |M| \frac{\varepsilon}{2|M|} + |B| \frac{\varepsilon}{2|B|} = \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We could also choose n so large that

$$|b_n - B| < \frac{\varepsilon}{M + |B|} \quad \text{and} \quad |a_n - A| < \frac{\varepsilon}{M + |B|},$$

then

$$\begin{aligned} |a_n b_n - AB| &\le M |b_n - B| + |B| |a_n - A| \\ &< M\left(\frac{\varepsilon}{M + |B|}\right) + |B|\left(\frac{\varepsilon}{M + |B|}\right) \\ &= (M + |B|)\left(\frac{\varepsilon}{M + |B|}\right) = \varepsilon. \end{aligned}$$

The choice of how to bound it depends on personal choice. After deciding, we write a formal proof.

<u>Theorem</u>: If $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B, then $\{a_n b_n\}_{n=1}^{\infty}$ converges to AB.

<u>Proof</u>: Let $\varepsilon > 0$ be given. Since $\{a_n\}$ is bounded, there exists M such that $|a_n| \leq M$ for all n. Choose N such that $|a_n - A| < \dfrac{\varepsilon}{M + |B|}$ and

$|b_n - B| < \dfrac{\varepsilon}{M + |B|}$. Then

$$
\begin{aligned}
|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\
&\leq |a_n||b_n - B| + |B||a_n - A| \\
&< M\left(\frac{\varepsilon}{M+|B|}\right) + |B|\left(\frac{\varepsilon}{M+|B|}\right) \\
&= M + |B|\left(\frac{\varepsilon}{M+|B|}\right) = \varepsilon.
\end{aligned}
$$

We can do the same for division of sequences. There is a careful investigation in the book, but here we present the "cleaned up" proof. First we need a small lemma.

<u>Lemma</u>: If $\{b_n\}_{n=1}^{\infty}$ converges to B and $B \neq 0$ then there exists $M > 0$ and an integer N such that if $n \geq N$ then $|b_n| \geq M$.

<u>Proof</u>: Set $\varepsilon = \dfrac{|B|}{2}$, note $\varepsilon > 0$.

For this particular ε, there exists N such that $n \geq N$ implies $|b_n - B| < \varepsilon$. Set $M = \frac{|B|}{2}$. Then for $n \geq N$, we have

$$|b_n| = |b_n - B + B| \geq |B| - |b_n - B| \geq |B| - \frac{|B|}{2}$$
$$= \frac{|B|}{2} = M.$$

Now we are ready to prove:

<u>Theorem</u>: Suppose $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B. If $B \neq 0$ and $b_n \neq 0$ for all n, then $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\frac{A}{B}$.

<u>Proof</u>: Let $\varepsilon > 0$. By the previous Lemma, there exists $M > 0$ and an integer N_1 such that $|b_n| \geq M$ for all $n \geq N_1$. Set

$$\varepsilon' = \frac{M\varepsilon}{1 + \left|\frac{A}{B}\right|}.$$

Now choose a positive integer N_2 such that $n \geq N_2$ implies $|a_n - A| < \varepsilon'$, and a positive integer N_3 such that $n \geq N_3$ implies $|b_n - B| < \varepsilon'$. Set $N = \max\{N_1, N_2, N_3\}$, so that for $n \geq N$ we have $|a_n - A| < \varepsilon'$, $|b_n - B| < \varepsilon'$, and $|b_n| \geq M$.

Now compute, for $n \geq N$:

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n B - b_n A}{b_n B}\right| = \left|\frac{a_n B - AB + AB - b_n A}{b_n B}\right|$$

$$\leq \left|\frac{a_n - A}{b_n}\right| + \frac{|A|\,|b_n - B|}{|b_n|\,|B|}$$

$$< \frac{1}{|b_n|}\varepsilon' + \frac{|A|}{|b_n|\,|B|}\varepsilon'$$

$$\leq \varepsilon'\left(\frac{1}{M}\left(1 + \frac{|A|}{|B|}\right)\right)$$

$$= \frac{M\varepsilon}{\left(1 + \left|\frac{A}{B}\right|\right)}\left(\frac{1}{M}\left(1 + \left|\frac{A}{B}\right|\right)\right) = \varepsilon.$$

Favourite quote from book:

" You have now been initiated into the exclusive club of epsilon pickers".

MATH 2080

§ 1.3 continued.

We can now use some familiar tricks for calculating limits of sequences.

Example: What does $\left\{\frac{n^3-1}{2n^3+n^2}\right\}_{n=1}^{\infty}$ converge to?

Solution: Rewrite this:

$$\frac{n^3-1}{2n^3+n^2} = \frac{1-1/n^3}{2+1/n}.$$

We saw already that $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 0. The sequence $\left\{\frac{1}{n^3}\right\}_{n=1}^{\infty}$ also converges to zero, being a product of $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ with itself 3 times.

Thus the sequence $\left\{1-\frac{1}{n^3}\right\}_{n=1}^{\infty}$ converges to 1, and $\left\{2+\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 2. Thus $\left\{\frac{1-1/n^3}{2+1/n}\right\}_{n=1}^{\infty}$ converges to $1/2$, being a quotient of these two.

So $\left\{\frac{n^3-1}{2n^3+n^2}\right\}_{n=1}^{\infty}$ converges to $1/2$.

Example: Consider the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}$.

Solution: We first multiply by the conjugate:

$$\sqrt{n+1}-\sqrt{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

But now observe that

$$0<\frac{1}{\sqrt{n+1}+\sqrt{n}}\leq\frac{1}{2\sqrt{n}}$$ (Since we replaced $\sqrt{n+1}+\sqrt{n}$ with the smaller term $2\sqrt{n}$)

and the sequence $\left\{\frac{1}{2\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to zero, so the sequence $\left\{\frac{1}{\sqrt{n+1}+\sqrt{n}}\right\}_{n=1}^{\infty}=\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}$ converges to zero as well, "by Exercise 9". (A squeeze theorem type result).

The previous example uses a fact we would like to investigate in more detail: If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converges to A and B respectively, and if $a_n \leq b_n$ for all n, do we get $A \leq B$? Yes!

<u>Theorem</u>: With $\{a_n\}_{n=1}^{\infty}$ converging to A and $\{b_n\}_{n=1}^{\infty}$ converging to B, if $a_n \leq b_n$ for all n then $A \leq B$.

<u>Proof</u>: Suppose $B < A$, then set $\varepsilon = \frac{A-B}{2} > 0$. Then there's an integer N_1 such that $n \geq N_1$ implies $A-\varepsilon < a_n < A+\varepsilon$, and there's an

integer N_2 such that $n \geq N_2$ implies $B - \varepsilon < b_n < B + \varepsilon$. Choose N to be $\max\{N_1, N_2\}$, then for $n \geq N$ we have

$$b_N < B + \varepsilon = A - \varepsilon < a_N \leq b_N$$

because $\varepsilon = \frac{A-B}{2}$

But this inequality is impossible, a contradiction. Thus we cannot have $B < A$, so $A \leq B$.

Example: If $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ both converge to some number A, and if $\{b_n\}_{n=1}^{\infty}$ converges to some number B, then

$$a_n \leq b_n \leq c_n \quad \text{for all } n$$

implies $A \leq B \leq A$, ie. $A = B$. This gives a sort of "squeeze theorem for sequences".

There is one weakness in the above squeeze theorem, however — with what we have done so far, we need to know that the middle sequence $\{b_n\}_{n=1}^{\infty}$ converges. This information may not be available in practice, and we'd like to be able to do a squeeze theorem in the absence of knowing a priori that $\{b_n\}_{n=1}^{\infty}$ converges.

Theorem: If $\{a_n\}_{n=1}^{\infty}$ converges to 0 and $\{b_n\}_{n=1}^{\infty}$ is bounded, then $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0.

Proof: Suppose M satisfies $|b_n| \leq M$ for all n. Now let $\varepsilon > 0$, and set $\varepsilon' = \frac{\varepsilon}{M} > 0$. Then there exists N such that $n \geq N$ implies

$$|a_n| = |a_n - 0| < \varepsilon \quad \text{(since } \{a_n\}_{n=1}^{\infty} \text{ converges to } 0\text{)}.$$

Then we compute:

$$\begin{aligned} |a_n b_n - 0| = |a_n b_n| = |a_n||b_n| &\leq |a_n| M \\ &< \varepsilon' M \\ &= \varepsilon. \end{aligned}$$

So $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0.

Example: Consider $\left\{\frac{1+(-1)^n}{2n}\right\}_{n=1}^{\infty}$. This is a product of the sequences $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty} = \{0, 1, 0, 1, 0, 1 \ldots\}$ and $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. The sequence $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}$ is bounded by 1, and $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 0. By the previous theorem, $\left\{\frac{1+(-1)^n}{2n}\right\}_{n=1}^{\infty}$ converges to 0 as well.

§ 1.4 Subsequences and monotone sequences

Definition: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and $\{n_k\}_{k=1}^{\infty}$ any sequence of positive integers with $n_1 < n_2 < n_3 < \ldots$ The sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{a_n\}_{n=1}^{\infty}$.

I.e: A subsequence is an infinite subset of a sequence, listed in their original order. ~~(but with new sub~~

E.g: The sequence $\left\{ \frac{1+(-1)^n}{2} \right\}_{n=1}^{\infty}$ is $\{0, 1, 0, 1, \ldots\}$.

The sequence $\{0, 0, 0, \ldots\}$ is a subsequence of the original, as is $\{1, 1, 1, \ldots\}$. The original sequence does not converge, but each of the subsequences do. Thus a divergent sequence can have convergent subsequences.

Question: Can a convergent sequence have divergent subsequences? If no, what do the subsequences converge to?

Example: Consider the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$. The sequences $\left\{\frac{1}{k^2}\right\}_{k=1}^{\infty}$, $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$ and $\left\{\frac{1}{2k}\right\}_{k=1}^{\infty}$ are all subsequences of this. According to our previous work, these sequences all converge to 0. So we guess:

Theorem: A sequence converges if and only if all of its subsequences converge, in which case they all converge to the same limit.