MATH 2080

§1.1 Sequences and convergence

Definition: A sequence is a function whose domain is $\mathbb{N}$.

Usually, instead of writing $f: \mathbb{N} \longrightarrow \mathbb{R}$ and then talking about the values $f(1), f(2), f(3), \ldots$ etc we write $a_1, a_2, a_3, \ldots$ so

$$f(1) = a_1$$
$$f(2) = a_2$$
$$\vdots$$
$$\text{etc.}$$

and we write $\{a_i\}_{i=1}^{\infty}$ for the collection of all such values. So, the sequence

$$f: \mathbb{N} \longrightarrow \mathbb{R}$$

given by $f(n) = n^2+1$ would be written as $\{n^2+1\}_{n=1}^{\infty}$ which is shorthand for the set

$$\{2, 5, 10, 17, 26, \ldots\}$$

Example: Write the first few terms of the sequence $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}$.

Solution: For $n=1, 2, 3, 4$ we get

$$\{\underset{n=1}{0}, \underset{n=2}{1}, \underset{n=3}{0}, \underset{n=4}{1}, 0, \ldots\} \text{ repeating indefinitely}$$

In general we see that

$$\frac{1+(-1)^n}{2} = 0 \text{ if } n \text{ odd}$$

$$\frac{1+(-1)^n}{2} = 1 \text{ if } n \text{ even.}$$

As n becomes very large, some sequences behave differently than others. For example,

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$

vs

$$\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}.$$

The values of the first sequence become closer and closer to 0, while the values of the second are sometimes near 0, sometimes not. More precisely: No matter the number x we are given, there's some point ~~where~~ past which all terms in the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ are smaller than x. This is not the case for $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}$, since the terms do not "stabilize" in any sense. The difference is that one of the sequences above <u>converges</u>, the other does not.

<u>Definition</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number A if and only if for each $\varepsilon > 0$

there is a positive integer N (depending on ε) such that for all $n \geq N$ we have $|a_n - A| < \varepsilon$.

<u>Example</u>: The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to $A = 0$.

To see this, suppose we are given $\varepsilon > 0$. Then by choosing $N > \frac{1}{\varepsilon}$, we can guarantee that whenever $n \geq N$ we've got:

$$a_n = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore for $n \geq N$ we have, with $A = 0$:

$$|a_n - A| = \left|\tfrac{1}{n}\right| \leq \frac{1}{N} < \varepsilon.$$

So it converges to 0.

<u>Example</u>: On the other hand, $\left\{\frac{1+(-1)^n}{2}\right\}_{n=1}^{\infty}$ does not converge to any real number A.

Suppose it did. Then for $\varepsilon = \frac{1}{2}$, there should be a number N such that $\left|\frac{1+(-1)^n}{2} - A\right| < \frac{1}{2}$ whenever $n \geq N$, but this is not possible! Here's why: The values of $\frac{1+(-1)^n}{2}$ alternate between 1 and 0. If the terms that are 1 are within $\frac{1}{2}$ of the number A, then the terms that are 0 must be <u>more</u> than $\frac{1}{2}$ away from A. So overall, not all terms in the sequence approach A.

Example: Show that $\left\{ \frac{2n+1}{3n-2} \right\}_{n=1}^{\infty}$ converges to $2/3$.

Proof: Let $\varepsilon > 0$. We want to find N such that $n \geq N$ gives $\left| \frac{2n+1}{3n-2} - \frac{2}{3} \right| < \varepsilon$.

This means we want $\left| \frac{3(2n+1) - 2(3n-2)}{9n-6} \right| < \varepsilon$

$$\Leftrightarrow \left| \frac{7}{9n-6} \right| < \varepsilon.$$

If $n \geq 1$ this is positive, so $\frac{7}{9n-6} < \varepsilon$.

Solving for n gives: $7 < (9n-6)\varepsilon$

$$\Leftrightarrow 7 + 6\varepsilon < 9n\varepsilon$$

$$\Leftrightarrow \frac{7+6\varepsilon}{9\varepsilon} < n.$$

So choose N to be bigger than $\frac{7+6\varepsilon}{9\varepsilon}$, then if $n \geq N$ we know $\left| \frac{2n+1}{3n-2} - \frac{2}{3} \right| < \varepsilon$. Thus the sequence converges to $2/3$.

Our study of the real numbers will be made easier by a few definitions:

<u>Definition</u>: A set Q of real numbers is a <u>neighbourhood</u> of $x \in \mathbb{R}$ if there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset Q$. We write <u>nbhd</u> (in this class)

Note in particular that $(x-\varepsilon, x+\varepsilon)$ is a neighbourhood of x for every $\varepsilon > 0$. This allows us to restate the definition of convergence.

<u>Lemma</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ converges to A iff each neighbourhood of A contains all but a finite number of terms of the sequence.

<u>Proof</u>: Suppose $\{a_n\}$ converges to A, and Q is a nbhd of A. Choose ε such that $(A-\varepsilon, A+\varepsilon) \subset Q$. Since $\{a_n\}_{n=1}^{\infty}$ converges to A, $\exists N$ such that

$$|a_n - A| < \varepsilon \text{ for all } n \geq N,$$

in other words $a_n \in (A-\varepsilon, A+\varepsilon) \subset Q$ for all $n \geq N$. Thus only $\{a_1, a_2, \ldots, a_{N-1}\}$ are possibly outside of Q.

On the other hand, suppose every nbhd of A contains all but a finite number of elements of $\{a_n\}_{n=1}^{\infty}$. ~~are~~ Let $\varepsilon > 0$. Then all but a finite number of $\{a_n\}_{n=1}^{\infty}$ are in $(A-\varepsilon, A+\varepsilon)$.

Let M be the largest integer such that $a_M \notin (A-\varepsilon, A+\varepsilon)$. Let $N = M+1$, then for all $n \geq N$ the elements $\{a_n\}_{n=N}^{\infty}$ are contained in $(A-\varepsilon, A+\varepsilon)$; meaning $|a_n - A| < \varepsilon$. Thus $\{a_n\}_{n=1}^{\infty}$ converges to A.

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Section 1.1 continued

Question: If $\{a_n\}_{n=1}^{\infty}$ converges to a number L, can it simultaneously converge to another number $M \neq L$?

Answer: No.

Theorem: Suppose that $\{a_n\}_{n=1}^{\infty}$ converges to both A and B. Then $A = B$.

Proof: For contradiction, suppose $A \neq B$, and that $A < B$. Let $\varepsilon = \frac{1}{2}(B-A)$, we will show that there's no N such that $|a_n - B| < \varepsilon$ <u>and</u> $|a_n - A| < \varepsilon$ for all $n \geq N$.

Suppose there is such an N, so we have

$$A - \varepsilon < a_n < A + \varepsilon \quad \text{for all } n \geq N.$$

But
$$\begin{aligned} A+\varepsilon = A + \tfrac{1}{2}(B-A) &= \tfrac{1}{2}(A+B) \\ &= B - \tfrac{1}{2}(B-A) \\ &= B - \varepsilon. \end{aligned}$$

This means that ~~since~~ when $a_n < A+\varepsilon$ we get $a_n < B-\varepsilon$ as well. So for all $n \geq N$, a_n is <u>not</u> in the interval $(B-\varepsilon, B+\varepsilon)$. This contradicts $\{a_n\}_{n=1}^{\infty}$ converging to B, and thus we must have $A = B$.

Sequences that converge behave differently in some fundamental ways than sequences which do not.

<u>Definition</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ is <u>bounded from above</u> if there is a real number M such that $a_n \leq M$ for all M, it is <u>bounded from below</u> if there is a real number P such that $P \leq a_n$ for all n. If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below, it's called <u>bounded</u>.

<u>Remark</u>: A bounded sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $P \leq a_n \leq M$ for some P and M, if we choose S such that $(P, M) \subset (-S, S)$, then we can see that this is equivalent to $|a_n| \leq S$ for all n.

<u>Example</u>: Some sequences fail to converge because they are not bounded, e.g. $\{n\}_{n=1}^{\infty}$. This sequence does not approach any limit A since the terms $\{1, 2, 3, \dots\}$ grow without bound.

Conversely, if $\{a_n\}_{n=1}^{\infty}$ <u>does</u> converge, we have a Theorem:

Theorem: If $\{a_n\}_{n=1}^{\infty}$ converges to A, then $\{a_n\}_{n=1}^{\infty}$ is bounded.

Proof: Suppose $\{a_n\}$ converges to A. Choose $\varepsilon = 1$, then $\exists N$ such that $n \geq N$ implies $a_n \in (A-1, A+1)$. Choose $S = \max\{a_1, a_2, \ldots, a_{N-1}, A+1\}$ and $P = \min\{a_1, a_2, \ldots, a_{N-1}, A-1\}$. Then $a_n \leq S$ for all n, because if $n < N$ then S is larger since it's a max of $\{a_1, \ldots, a_{N-1}\}$, whereas if $n \geq N$ then $a_n \leq S$ since $S \geq A+1$. Similarly for $a_n \geq P$, thus a_n is bounded.

Example: Consider the sequence

$a_1 = 1$, $a_2 = 1 + \frac{1}{2}$, $a_3 = 1 + \frac{1}{2} + \frac{1}{3}$, and in general

$$a_n = 1 + \sum_{i=2}^{n} \frac{1}{i}.$$

Does this sequence converge or diverge?

Consider the term a_{2^n}, and group the terms according to powers of 2:

$$a_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \underset{\substack{\uparrow \\ \text{power of } 2}}{\frac{1}{4}}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \underset{\substack{\uparrow \\ \text{power of } 2}}{\frac{1}{8}}\right) + \cdots + \left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots + 2^{n-1}\left(\frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$$

$$= 1 + \frac{n}{2}.$$

So this sequence cannot converge, because the terms a_{2^n} are bigger than $1 + n/2$ for each n — thus $\{a_n\}_{n=1}^{\infty}$ is unbounded.

<u>Definition</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ is called <u>convergent</u> if it converges, and <u>divergent</u> if it does not converge.

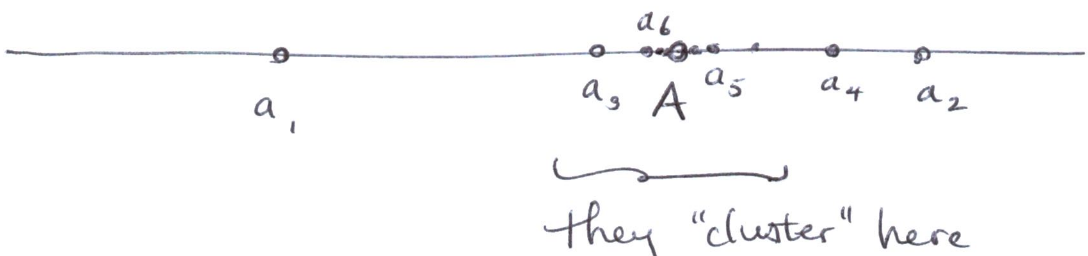
<u>Summary</u>: If a sequence is convergent, it's bounded an converges to a unique number L. If it's not convergent it's divergent, one way a sequence can diverge is by being unbounded.

If $\{a_n\}_{n=1}^{\infty}$ converges to L, L is called <u>its limit</u>

<u>Section 1.2</u> Cauchy Sequences.

Next we wish to investigate another condition similar to convergence, it will be called the <u>Cauchy condition</u>.

The idea is this: Suppose $\{a_n\}_{n=1}^{\infty}$ converges, say to A. Then as $n \to \infty$, the points a_n are clustering very close to A:



However, at the same time that the terms a_n cluster close to A, the also cluster <u>close to one another</u>.

This is the Cauchy condition:

<u>Definition</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ is Cauchy if for every $\varepsilon > 0$ there exists an $N \geq 0$ such that whenever $n, m \geq N$ then $|a_n - a_m| < \varepsilon$.

<u>Remark</u>: This means that, given $\varepsilon > 0$, we can ensure that points in the sequence $\{a_n\}_{n=1}^{\infty}$ are less than ε apart if we simply look far enough out in the sequence.

<u>Example</u>: Let $a_n = \frac{1}{n^2}$. Then $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. To see this, let $\varepsilon > 0$ be given. We need to find N such that $m, n \geq N$ implies

$$\left| \frac{1}{m^2} - \frac{1}{n^2} \right| < \varepsilon.$$

Choose $N > \frac{2}{\varepsilon}$. Then $m, n \geq N > \frac{2}{\varepsilon}$ gives

$$\frac{1}{n} < \frac{\varepsilon}{2} \text{ and } \frac{1}{m} < \frac{\varepsilon}{2}, \text{ so}$$

$$\left|\frac{1}{n^2}-\frac{1}{m^2}\right| \leq \left|\frac{1}{n^2}\right|+\left|\frac{1}{m^2}\right| \quad \text{(triangle inequality)}$$

$$= \frac{1}{n^2}+\frac{1}{m^2}$$

since $n, m \geqslant 1$

$$\leq \frac{1}{n}+\frac{1}{m}$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

So $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ is a Cauchy sequence.

§ 1.2 Cauchy Sequences continued.

The idea of a Cauchy sequence is that terms in the sequence "get close to one another", whereas the idea of a sequence converging to a value $A \in \mathbb{R}$ is that terms in the sequence get close to A. Surprisingly, these are equivalent.

Theorem: Every convergent sequence is Cauchy.

Proof: Suppose $\{a_n\}_{n=1}^{\infty}$ converges to A, and choose $\varepsilon > 0$. Then there's N such that $n \geq N$ implies $|a_n - A| < \varepsilon/2$. Now if $m \geq N$, we have $|a_m - A| < \varepsilon/2$ as well. Therefore

$$|a_n - a_n| = |a_n - A + A - a_m|$$

$$\leq |a_n - A| + |a_m - A| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $\{a_n\}_{n=1}^{\infty}$ is Cauchy.

We next want to show the other direction, ~~we~~ namely that Cauchy sequences are convergent. This is quite tricky, so we will need to do one or two classes of prep work.

First, we note that like convergent sequences, Cauchy sequences are bounded.

Theorem: Every Cauchy sequence is bounded.

Proof: On next assignment.

Next, we need a definition to make our discussion smoother.

Definition: Let S be a set of real numbers. A real number $A \in \mathbb{R}$ is an accumulation point of S if every nbhd of S contains infinitely many points of S.

There is another, sometimes useful way of saying this:

Lemma: Let S be a set of real numbers. Then $A \in \mathbb{R}$ is an accumulation point of S iff each neighbourhood of A contains members of S different from A.

Proof: Assume A is an accumulation point of S. Then every nbhd of A, as it contains infinitely many points of S, must contain some point of S other than A.

On the other hand, if A is not an accumulation point, then some nbhd of A only contains finitely many points of S, say $\{s_1, s_2, \dots, s_n\}$. Suppose that s_i is the closest to A (which is not equal to A). Set $\varepsilon = |s_i - A|$. Then $(A-\varepsilon, A+\varepsilon)$ contains no elements of S aside from possibly A (if $A \in S$). This proves the lemma.

Remark: The limit of a sequence $\{a_n\}_{n=1}^{\infty}$ is not always an accumulation point of the set $\{a_n\}_{n=1}^{\infty}$. The sequence $\{1, 1, 1, \dots\}$ (all one's) clearly converges to $A = 1$, but the set $\{1, 1, \dots\}$ has no accumulation points.

Our goal now: Describe which sets must have accumulation points, and then use our description to show that Cauchy sequences always have accumulation points — as long as the sequence is not eventually constant, like the sequence of 1's above. Then show that the accumulation point we found is actually the limit of the sequence.

Theorem (Bolzano-Weierstrass Theorem). Every bounded infinite set of real numbers has at least one accumulation point.

Proof: We apply a recursive procedure to find such a point.

Let S be a bounded and infinite subset of $\mathbb{R}$, say $S \subset [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$.

Let $\alpha_1 = \frac{\beta - \alpha}{2}$, then at least one of $[\alpha, \alpha_1]$ or $[\alpha_1, \beta]$ contains infinitely many points of S, so choose the interval with this property and call it $[a_1, b_1]$. Now if $\alpha_2 = \frac{b_1 - a_1}{2}$, at least one of $[a_1, \alpha_2]$ and $[\alpha_2, b_1]$ contains infinitely many points of S, choose such an interval and call it $[a_2, b_2]$. Repeating, we arrive at intervals $[a_n, b_n]$ with:

(i) $b_n - a_n = \frac{1}{2^n}(\beta - \alpha)$ (because at each step we cut the previous interval in half)

(ii) $[a_n, b_n]$ contains infinitely many points of S.

(iii) $[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset \dots \subset [\alpha, \beta]$ for all n.

Look at the collection of lower endpoints $Q = \{a_n \mid n = 1, 2, \dots\}$, it's bounded so it has a supremum $t = \sup Q$. We will show t is the accumulation point we want, by showing every neighbourhood of t contains infinitely many points of S.

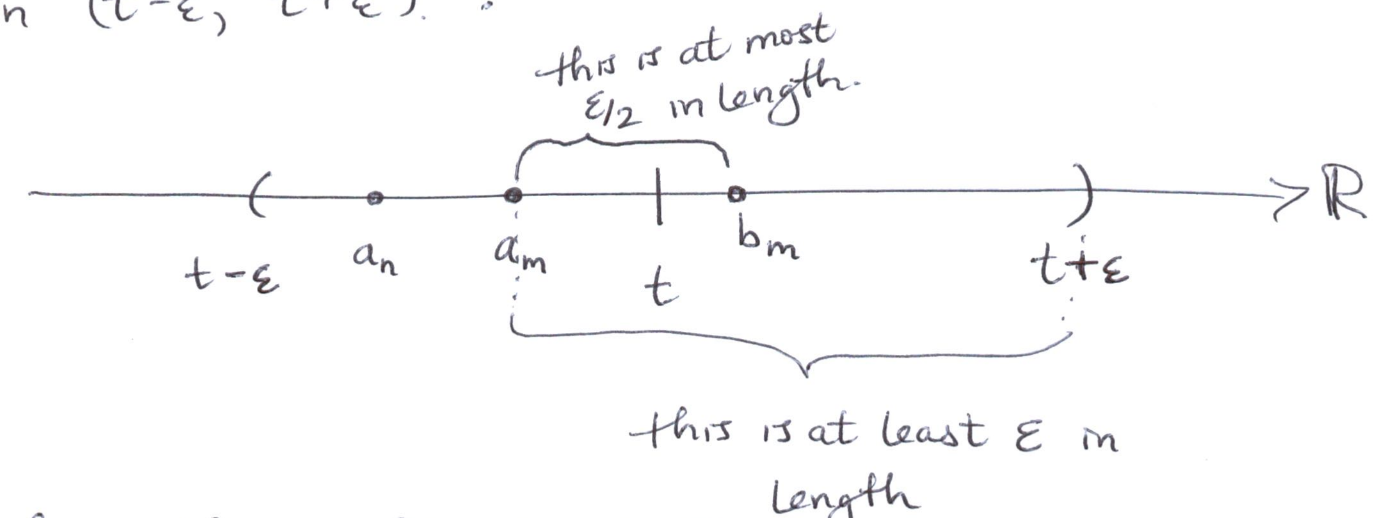
So let P be a neighbourhood of P, and chose $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \subset P$. Since $t - \varepsilon$ is not an upper bound for Q, there's a_m with

$$t - \varepsilon < a_n \leq t,$$ and in fact if $m > n$ then

$$t - \varepsilon < a_n < a_m \leq t.$$

Choose $m > n$ so large that $2^{-m}(\beta - \alpha) < \varepsilon/2$

Then the interval $[a_m, b_m]$ is of length $\frac{\beta-\alpha}{2^m}$, which is less than $\varepsilon/2$. Since $t - \varepsilon < a_m \leq t$, this means the interval $[a_m, b_m]$ is contained in $(t-\varepsilon, t+\varepsilon)$:



Thus, the neighbourhood P of t contains $[a_m, b_m]$, which contains infinitely many points of S. So t is an accumulation point of S.

Now we are ready to prove: Every Cauchy sequence is convergent, and we'd like to apply the Bolzano-Weierstrass theorem.

<u>Theorem</u>: Every Cauchy sequence is convergent.

<u>Proof</u>. We have two cases. First, suppose that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. ~~By~~

And further suppose that $\{a_1, a_2, \dots, a_n, \dots\}$ is actually a finite set. Say

$$\{a_1, a_2, \dots\} = \{s_1, s_2, \dots, s_r\}.$$

Choose $\varepsilon = \min\{|s_i - s_j| \mid i \neq j,\ i,j \in \{1,\dots,r\}\}$

i.e. ε is the minimum distance between two distinct points in the set $\{s_1, s_2, \dots, s_r\}$. Then there's an $N \in \mathbb{N}$ such that for $m, n \geq N$ $|a_m - a_n| < \varepsilon$. But if a_m and a_n are closer than the minimum distance between points in $\{a_1, a_2, \dots\}$, then they must be equal. So for $m, n \geq N$, $a_m = a_n$. Thus the sequence is constant past the term a_N, thus convergent.

On the other hand, suppose $\{a_1, a_2, \dots\}$ is infinite. By our preceding work, $\{a_1, a_2, \dots\}$ is bounded since Cauchy sequences are bounded. Thus by the Bolzano-Weierstrass theorem, $\{a_n\}_{n=1}^{\infty}$ has an accumulation point — call it a. We'll prove that $\{a_n\}$ converges to a.

Let $\varepsilon > 0$. Then the nbhd $(a - \varepsilon/2, a + \varepsilon/2)$ contains infinitely many terms in the sequence $\{a_n\}_{n=1}^{\infty}$. Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, there exists an N such that $m, n \geq N$ gives $|a_m - a_n| < \varepsilon/2$. Also since $(a - \varepsilon/2, a + \varepsilon/2)$ contains infinitely many of the a_n's, there's an $n_0 \geq N$

such that $a_{n_0} \in (a - \varepsilon/2, a + \varepsilon/2)$.

Now for all $n \geq n_0$, we calculate

$$|a_n - a| \leq |a_n - a_0| + |a_{n_0} - a|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $\{a_n\}_{n=1}^{\infty}$ converges to a.

So a sequence is convergent iff it is Cauchy. This allows us to prove that a sequence is convergent while having no idea what its limit may be.