MATH 2080

We are now in a position to attack one of the most significant results in a first analysis course: The intermediate value theorem. We begin with a special case:

<u>Theorem</u>: (Bolzano's Theorem).

Suppose $f: [a,b] \longrightarrow \mathbb{R}$ is continuous and that $f(a)$ and $f(b)$ have opposite signs. Then there exists $z \in (a,b)$ with $f(z) = 0$.

<u>Proof</u>: We'll do the case $f(a) < 0$ and $f(b) > 0$, since the other case is very similar.

Now set $a_1 = \dfrac{a+b}{2}$.

If $f(a_1) = 0$ then $a_1 = z$ proves the theorem. If $f(a_1) > 0$, set $I_1 = (a, \underset{\overset{\|}{a_1}}{b_1})$ and if $f(a_1) < 0$ set $I_1 = (a_1, \underset{\overset{\|}{b_1}}{b})$. (ie rename $a_1 = b_1$)

So if $I_0 = (a,b)$ then $I_1 \subset I_0$ and I_1 is half the length of I_0. Repeat this procedure to create a sequence of intervals $I_n = (a_n, b_n)$ with:

(i) $I_{n+1} \subset I_n$ for all n

(ii) $b_n - a_n = \dfrac{1}{2^n}\, b - a$

(iii) $f(a_n) < 0 < f(b_n)$.

Define a sequence $\{c_n\}_{n=1}^{\infty}$ by $c_{2n} = a_n$ and $c_{2n+1} = b_n$. By (ii) above, the sequence $\{c_n\}_{n=1}^{\infty}$ is Cauchy and so converges to some $c \in [a,b]$. Since $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are subsequences of $\{c_n\}_{n=1}^{\infty}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c .$$

Then by continuity of f,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(c)$$

But $f(a_n) < 0$ and $f(b_n) > 0$ for all n. Thus $\lim_{n\to\infty} f(a_n) \leq 0$ and $\lim_{n\to\infty} f(b_n) \geq 0$, so

$$0 \leq f(c) \leq 0 \Rightarrow f(c) = 0$$

<u>Remark</u>: Dissecting the proof above, we ask: Why would this theorem only hold for functions $f : [a,b] \longrightarrow \mathbb{R}$, and not more general functions $f : D \longrightarrow \mathbb{R}$? Well, the argument would fail if there were any point between a and b that were not in the domain of f. So we need this property to ensure the proof works.

<u>Definition</u>: A set $A \subset \mathbb{R}$ is <u>connected</u> if whenever a and b are in A, and $a<c<b$, then $c \in A$.

This is certainly the correct definition to make, but what kinds of sets are captured by this condition?

<u>Theorem</u>: Let A be a connected subset of $\mathbb{R}$. Then A is one of the following:

(i) $\{x \mid x<a\}$, $\{x \mid x>a\}$, $\{x \mid x \leq a\}$, $\{x \mid x \geq a\}$

(ii) (a,b), $(a,b]$, $[a,b)$ or $[a,b]$

(iii) $\mathbb{R}$.

<u>Proof</u>: Each of the sets listed above is connected. We need to show that any connected set $A \subset \mathbb{R}$ falls into one of these categories.

So let A be a connected set. If $A=\emptyset$, then $A=(a,a)$ for any $a \in \mathbb{R}$ and so is in category (ii). So suppose A is nonempty.

If A is not bounded above or below, then $A=\mathbb{R}$ and so we're in (iii). If A is bounded from above but not below, then set $a=\sup A$. If $a \in A$ then we can argue $A=\{x \in \mathbb{R} \mid x \leq a\}$ while if $a \notin A$ then $A=\{x \in \mathbb{R} \mid x<a\}$. On the other hand if A is bounded below but not above, then $a=\inf A$ gives the possibilities

$A = \{x \in \mathbb{R} \mid x \geq a\}$ or $A = \{x \in \mathbb{R} \mid x > a\}$.

Last, if A is bounded both above and below set $a = \inf A$ and $b = \sup A$. Then a careful case argument gives $A = [a,b]$ or (a,b) or $[a,b)$ or $(a,b]$.

Theorem: Suppose $f: A \to \mathbb{R}$ is continuous with A connected. Then if $a<b$ with $a,b \in A$ and $f(a)$, $f(b)$ having opposite signs, there exists $c \in (a,b)$ with $f(c) = 0$.

Proof: Same proof as before.

From this, we can easily prove the Intermediate Value Theorem.

Theorem: Suppose $f: A \to \mathbb{R}$ is continuous and A is connected. Suppose also that $a<b$ with $a,b \in A$ and that $y \in \mathbb{R}$ is between $f(a)$ and $f(b)$. Then there exists $c \in (a,b)$ with $f(c) = y$.

Proof: Define $g: A \to \mathbb{R}$ by $g(x) = f(x) - y$. Then g is continuous, and since y is between $f(a)$ and $f(b)$, the numbers $g(a)$ and $g(b)$ have opposite signs. Thus by Bolzano's theorem there is $c \in (a,b)$ with $g(c) = 0$, ie $f(c) - y = 0$ so $f(c) = y$.

Consequences: Every continuous function maps intervals to intervals.

Theorem: Suppose $f:[a,b]\longrightarrow\mathbb{R}$ is continuous. Then there exist c,d such that $f([a,b])=[c,d]$.

Proof: Since f attains a max and min on $[a,b]$, there exist $x_1, x_2\in[a,b]$ such that $f(x_1)\le f(x)\le f(x_2)$ for all $x\in[a,b]$. Let $c=f(x_1)$ and $d=f(x_2)$. By the Intermediate value theorem, given any $y\in[c,d]$ there exists $x\in[a,b]$ such that $f(x)=y$. Thus $f([a,b])=[c,d]$.

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§3.4

As a final consequence of compactness, we have:

Theorem: Let $f: A \to \mathbb{R}$ be continuous and one-to-one, and suppose A is connected. Then f is monotone.

Proof: Suppose f is 1-1 and continuous but not monotone. Then either

(i) There exist $x, y, z \in A$ with $x < y < z$ and $f(x) < f(y)$ and $f(z) < f(y)$

or

(ii) There exist $x, y, z \in A$ with $x < y < z$ and $f(x) > f(y)$ and $f(z) > f(y)$.

We will deal only with case (ii). In that case, Suppose $f(y) < f(z) < f(x)$. Then by the intermediate value theorem, there is c in $[x, y]$ such that $f(c) = f(z)$. But $x < y < z$ means $c \neq z$, so $f(c) = f(z)$ contradicts f being one-to-one. Similarly if $f(y) < f(x) < f(z)$, then the intermediate value theorem says there is a c with $c \in [y, z]$ and $f(c) = f(x)$. But $x < y < z$ means $c \neq x$, so this again contradicts f being 1-1.

Case (i) is proved similarly.

Chapter 4 Differentiation.

We define the derivative as usual:

Definition: If $f: D \to \mathbb{R}$ and x_0 is an accumulation point of D, ~~then~~ and $x_0 \in D$. If

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists then f is said to be differentiable at x_0 (and its derivative is $f'(x_0)$).

Recall now that if f is differentiable at x_0, then f is continuous at x_0:

Theorem: If $f: D \to \mathbb{R}$ is differentiable at x_0 (meaning $x_0 \in D$ and x_0 is an acc pt of D) then f is continuous at x_0.

Proof: We have, for any $x \in D$:

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$

$$\Rightarrow \lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$

Assuming $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ exists, this gives:

$$\lim_{x\to x_0} f(x) - \lim_{x\to x_0} f(x_0) = f'(x_0)\cdot 0 = 0$$

$$\Rightarrow \lim_{x\to x_0} f(x) = f(x_0), \text{ so } f \text{ is continuous.}$$

All the usual rules of derivatives (sum, product, chain rule, etc) follow as in MATH 1500.

§4.3 Rolle's theorem and the Mean Value Theorem.

Calculus is supposed to be used for finding maxes/mins, solving real-world problems (sometimes). So let's touch on that to end the course.

Definition: Let $f: D \to \mathbb{R}$. A point $x_0 \in D$ is a relative maximum (minimum) of $f(x)$ if and only if there is a nbhd Q of x_0 such that if $x \in Q \cap D$, then $f(x) \le f(x_0)$.

Theorem: Suppose $f: [a,b] \to \mathbb{R}$ and suppose that f has either a relative maximum or a relative minimum at $x_0 \in (a,b)$. If $f(x)$ is differentiable at x_0, then $f'(x_0) = 0$.

Proof: Assume f has a relative maximum at $x_0 \in (a,b)$.

Then there exists $\delta > 0$ such that for all x in $(x_0 - \delta, x_0 + \delta)$ we have $f(x) \leq f(x_0)$. Consider any sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_0 - \delta < x_n < x_0$ for all n. Then since f is differentiable at x_0,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists, meaning } \left\{ \frac{f(x_n) - f(x_0)}{x_n - x_0} \right\}_{n=1}^{\infty}$$

converges (to $f'(x_0)$). But $f(x_n) \leq f(x_0)$ and $x_n \leq x_0$ for all n, so $\dfrac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0$ for all n.

So $f'(x_0) \geq 0$. On the other hand if $\{y_n\}_{n=1}^{\infty}$ is a sequence converging to x_0 with $x_0 < y_n < x_0 + \delta$ then $\left\{ \dfrac{f(y_n) - f(x_0)}{y_n - x_0} \right\}_{n=1}^{\infty}$ also converges to $f'(x_0)$, but

$\dfrac{f(y_n) - f(x_0)}{y_n - x_0} \leq 0$ for all n. Thus $f'(x_0) \leq 0$.

<u>In</u> conclusion, $f'(x_0) = 0$. |

Thus we can prove

<u>Rolle's Theorem</u>: If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and f is differentiable on (a,b) then $f(a) = f(b) = 0$ implies that there exists $c \in (a,b)$ with $f'(c) = 0$.

Proof: If $f(x)=0$ for all $x \in [a,b]$, then done.

Otherwise suppose $f(x) \neq 0$ for some $x \in [a,b]$. Then f attains a max and a min on $[a,b]$ since $[a,b]$ is compact. Say $f(x_1)$ is the max and $f(x_2)$ is the min. Then at least one of $f(x_1)$ or $f(x_2)$ is nonzero, and thus at least one of x_1 or x_2 is not equal to a or b. By the previous theorem, f' is zero there.