§3.3 Open, closed and compact sets.

Recall that a set X is <u>closed</u> iff every accumulation point of X belongs to X. A set is <u>open</u> if the set contains a neighbourhood of each of its points.

<u>Theorem</u>: A set $E \subset \mathbb{R}$ is closed if and only if $\mathbb{R} \setminus E$ is open.

<u>Proof</u>: Suppose E is closed. Choose $x_0 \in \mathbb{R} \setminus E$. Then x_0 is <u>not</u> an accumulation point of E, so there exists a neighbourhood Q of x_0 that contains no points of E. Thus $x_0 \in Q \subset \mathbb{R} \setminus E$, so $\mathbb{R} \setminus E$ is open.

On the other hand suppose $\mathbb{R} \setminus E$ is open. Let $x_0 \in \mathbb{R}$ be an accumulation point of E. If $x_0 \in \mathbb{R} \setminus E$, then since $\mathbb{R} \setminus E$ is open there exists a nbhd Q of x_0 with $x_0 \in Q \subseteq \mathbb{R} \setminus E$, but this contradicts x_0 being an accumulation point of E. So we must have $x_0 \in E$.

<u>Remark</u>: Sets (a, b) are open, and sets $[a, b]$ are closed. In fact if (a_i, b_i) with $i \in I$ is some collection of intervals, then

$$\bigcup_{i \in I} (a_i, b_i) \text{ is open. (check this!)}$$

On the other hand, if $[a_i, b_i]$ with $i \in I$ are closed, $\bigcup_{i \in I} [a_i, b_i]$ could be open, closed, or neither.

For example if $[a_i, b_i] = [-1/i, 1/i]$ then

$$\bigcup_{i=1}^{\infty} [a_i, b_i] = [-1, 1] \qquad \text{(closed)}$$

while if $[a_i, b_i] = [-1 + 1/i, 1 - i/n]$ then

$$\bigcup_{i=1}^{\infty} [a_i, b_i] = (-1, 1) \qquad \text{(prove this to yourself!)}$$
(open)

and if $[a_i, b_i] = [-1 + 1/i, 1/i]$ then

$$\bigcup_{i=1}^{\infty} [a_i, b_i] = (-1, 1] \qquad \text{(neither)}.$$

Similarly, if $[a_i, b_i]$ are closed then

$\bigcap_{i=1}^{\infty} [a_i, b_i]$ is closed, but

$\bigcap_{i=1}^{\infty} (a_i, b_i)$ could be closed, open, or neither.

Now return to discussing uniform continuity, and use these ideas to approach a new definition:

Suppose $f : E \longrightarrow \mathbb{R}$ and f is continuous. Let $\varepsilon > 0$. When can we find $\delta > 0$ such that $x, y \in E$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$?

Well, for each fixed $x \in E$ there's $\delta_x > 0$, by continuity, so that $|x-y| < \delta_x \Rightarrow |f(x)-f(y)| < \varepsilon$. So to try and find a $\delta > 0$ that works for <u>all</u> x's simultaneously, we could set

$$\delta = \inf\{\delta_x \mid x \in E\}.$$

But this may fail since, even though all the δ_x's are positive, $\delta = \inf\{\delta_x \mid x \in E\}$ may be zero (and we seek $\delta > 0$). How could we guarantee that δ defined as above is positive?

Well, if E is finite then

$$\delta = \inf\{\delta_x \mid x \in E\} = \max\{\delta_x \mid x \text{ in some finite set}\} > 0.$$

If E is <u>not</u> finite, then for a fixed x, consider the set ~~the~~ $(x-\delta_x, x+\delta_x) \cap E$. If y_1, y_2 are in this set, then $|y_1 - x| < \delta_x$ and $|y_2 - x| < \delta_x$ so $|y_1 - y_2| < 2\delta_x$ and

$$|f(y_1) - f(y_2)| \le |f(y_1) - x| + |f(y_2) - x|$$
$$< \varepsilon + \varepsilon = 2\varepsilon.$$

So if we could cover E with finitely many of these sets, ie. if there are $\{x_1, \dots, x_n\}$ such that

$$E \subset \bigcup_{i=1}^{n} (x_i - \delta_{x_i}, x_i + \delta_{x_i})$$

then choosing $\delta = \min\{\delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_n}\}$ would make f uniformly continuous on E. We give a special name to the sets where such a choice of open sets $(x_1 - \delta_{x_1}, x_1 + \delta_{x_1})$, $(x_2 - \delta_{x_2}, x_2 + \delta_{x_2})$, ... is always possible.

Definition: A set $E \subseteq \mathbb{R}$ is compact iff for every family $\{G_\alpha\}_{\alpha \in A}$ of open sets with

$$E \subset \bigcup_{\alpha \in A} G_\alpha$$

there exist a finite collection $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$.

There is an easier way of saying this. If $\{G_\alpha\}_{\alpha \in A}$ is some collection of sets with $E \subset \bigcup_{\alpha \in A} G_\alpha$, then $\{G_\alpha\}_{\alpha \in A}$ is called a cover of E. If all the G_α's are open, it's called an open cover. If $B \subset A$ then $\{G_\alpha\}_{\alpha \in B}$ is a subcover of $\{G_\alpha\}_{\alpha \in A}$, and if B is finite it's called a finite subcover. Then:

A set E is compact iff every open cover of E has a finite subcover

Examples of non-compact sets:

- Set $E = (0,1]$. Set $G_n = (\frac{1}{n}, 2)$ for each $n \in \mathbb{N}$.

Then
$$(0,1] \subset \bigcup_{n \in \mathbb{N}} G_n = \bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2), \quad \text{because}$$
for each $x \in (0,1]$ there's $\frac{1}{n} < x$ so that $x \in (\frac{1}{n}, 2)$.
However $\{G_n\}_{n=1}^{\infty}$ has no finite subcover. For if we take a finite union
$$G_{n_1} \cup G_{n_2} \cup \dots \cup G_{n_k} \quad (\text{some } k > 0).$$
then set $n_0 = \max\{n_1, n_2, \dots, n_k\}$, and observe
$$\bigcup_{i=1}^{k} G_{n_i} \subset (\tfrac{1}{n_0}, 2)$$
and $(\frac{1}{n_0}, 2)$ does <u>not</u> contain $(0,1]$.

- The set $\mathbb{R}$ is not compact. Set $G_n = (-n, n)$, for each $n \in \mathbb{N}$. Then $\mathbb{R} \subset \bigcup_{n=1}^{\infty} G_n$, but it's not contained in any finite subcover $\bigcup_{i=1}^{k} G_{n_i}$, because $\bigcup_{i=1}^{k} G_{n_i}$ only contains real numbers x with
$$|x| < \max\{n_1, n_2, \dots, n_k\}.$$

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Which sets are compact?

Theorem: (Heine-Borel) A set $E \subseteq \mathbb{R}$ is compact if and only if E is closed and bounded.

Proof: First do compact $\Rightarrow$ closed and bounded.

Let $E \subseteq \mathbb{R}$ be compact, but not closed. Then there exists an accumulation point x of E that is not in E. For each $y \in E$, choose an open neighbourhood Q_y of x that does not intersect some open neighbourhood P_y of y. E.g. we could choose

$$Q_y = \left(x - \frac{|y-x|}{2}, x + \frac{|y-x|}{2}\right)$$

$$\text{and } P_y = \left(y - \frac{|y-x|}{2}, y + \frac{|y-x|}{2}\right)$$

Then $\{P_y\}_{y \in E}$ is an open cover of E. However there is no finite subcover, because $\bigcup_{i=1}^{n} P_{y_i}$ will always be disjoint from $\bigcap_{i=1}^{n} Q_{y_i}$ for any $\{y_1, \ldots, y_n\}$. Thus if $E \subset \bigcup_{i=1}^{n} P_{y_i}$ then $\bigcap_{i=1}^{n} Q_{y_i}$ is an open neighbourhood of x containing no points of E, a contradiction. So compact $\rightarrow$ closed.

Now we show compact $\rightarrow$ bounded. Suppose $E \subseteq \mathbb{R}$ is compact ~~by~~ but not bounded. Then

$$\bigcup_{n=1}^{\infty} (-n, n)$$

is an open cover of E which can have no finite subcover since E is unbounded. So compact $\Rightarrow$ closed and bounded.

Now closed + bounded implies compact.

First we prove: Any closed interval $[a,b]$ is compact.
Let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of $[a,b]$. Then there's $\alpha_0 \in A$ such that $a \in G_{\alpha_0}$, since G_{α_0} is open there's $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subset G_{\alpha_0}$. But then $[a, a+\frac{\varepsilon}{2}]$ is covered by the single open set G_{α_0}.

Set $B = \{x \in [a,b] \mid [a,x] \text{ can be covered by a finite subcover of } \{G_\alpha\}_{\alpha\in A}\}$.

The set B is nonempty since $a+\varepsilon/2 \in B$ and it's bounded since it only contains x with $a \le x \le b$. Set $z = \sup B$, then $a+\varepsilon/2 \le z \le b$, so there's $\alpha_1 \in A$ such that $z \in G_{\alpha_1}$. Since G_{α_1} is open, there's $\delta > 0$ such that $(z-\delta, z+\delta) \subset G_{\alpha_1}$. But then $[a, z-\delta/2]$ can be covered by a finite subcover of $\{G_\alpha\}_{\alpha\in A}$ since $z-\delta/2 \in B$, and so $[a, z+\delta/2]$ can <u>also</u> be covered by a finite subcover by simply adding G_{α_1} to the subcover that covers $[a, z-\delta/2]$. But then $z+\delta/2 \in B$, contradicting $z = \sup B$, unless $z = b$ and so $[a,b]$ is compact.

Now suppose $E \subseteq \mathbb{R}$ is closed and bounded. Choose a, b such that $E \subseteq [a,b]$. Let $\{G_\alpha\}_{\alpha\in A}$ be any open cover of E. Then $\{G_\alpha\}_{\alpha\in A} \cup \cancel{\{\mathbb{R}\setminus[a,b]\}}$ $\{\mathbb{R}\setminus E\}$ is an open cover of $[a,b]$.

Thus there's a finite $B \subset A$ such that $\{G_\alpha\}_{\alpha\in B} \cup \{\mathbb{R}\setminus E\}$ covers $[a,b]$. Since no points of E are in $\mathbb{R}\setminus E$, the finite subcover $\{G_\alpha\}_{\alpha\in B}$ must cover E. So E is compact.

Now if you recall, we motivated our discussion of compactness by talking about which sets E would cause continuous functions $f: E \to \mathbb{R}$ to be <u>uniformly</u> continuous.

<u>Theorem</u>: Let $f: D \to \mathbb{R}$ and suppose D is closed and bounded (compact). Then if f is continuous, it's uniformly continuous.

<u>Proof</u>: Let $\varepsilon > 0$. By continuity, for each $x \in D$ there's $\delta_x > 0$ such that $|x-y| < \delta_x \Rightarrow |f(x)-f(y)| < \varepsilon/2$ whenever $y \in D$.

The sets $\left\{\left(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}\right)\right\}_{x\in D}$ form an open cover of D, so there's a finite subcover. Say that it's the points $x_1, \ldots, x_n$ that give

$$D \subset \bigcup_{i=1}^{n} \left(x_i - \frac{\delta_{x_i}}{2},\ x_i + \frac{\delta_{x_i}}{2} \right).$$

Set $\delta = \min\{\delta_{x_i}/2 \mid i = 1, \ldots, n\}$. Let $x, y \in D$ be given, and suppose $|x - y| < \delta$. Then there's an x_i such that $x \in \left(x_i - \frac{\delta_{x_i}}{2},\ x_i + \frac{\delta_{x_i}}{2} \right)$, so that $|x - y| < \delta \le \frac{\delta_{x_i}}{2}$ implies

$$|y - x_i| \le |y - x| + |x - x_i| < \delta + \delta_{x_i}/2 < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}$$

Therefore

$$|f(x) - f(y)| \le \underbrace{|f(x) - f(x_i)|}_{\text{since } |x - x_i| < \frac{\delta_{x_i}}{2}} + \underbrace{|f(y) - f(x_i)|}_{\text{since } |y - x_i| < \frac{\delta_{x_i}}{2}.} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So continuous functions are <u>uniformly continuous</u> on compact sets.

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§3.4 Properties of continuous functions.

In these notes we give an alternative proof of Theorems 3.9 and Theorem 3.10 from the text. Here is our approach:

First we observe the relationship between continuous functions and open sets. Recall that if $f: X \longrightarrow Y$ is a function and $A \subset Y$, then

$$f^{-1}(A) = \{ x \in A \mid f(x) \in A \}.$$

<u>Theorem</u>: Suppose $f: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and that $U \subseteq \mathbb{R}$ is open. Then $f^{-1}(U)$ is open.

<u>Proof</u>: Let $x_0 \in f^{-1}(U)$ be given. Then $f(x_0) \in U$, so there's an $\varepsilon > 0$ such that $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U$ since U is open. But f is continuous, so corresponding to this ε there's a δ such that if $x \in (x_0 - \delta, x_0 + \delta)$ then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U$. But then every $x \in (x_0 - \delta, x_0 + \delta)$ lies in $f^{-1}(U)$, so x_0 has a neighbourhood $Q = (x_0 - \delta, x_0 + \delta)$ entirely in $f^{-1}(U)$. So $f^{-1}(U)$ is open.

Now we provide an alternative proof of Theorem 3.10.

Theorem: Let $f: E \to \mathbb{R}$ be continuous with E compact. Then $f(E)$ is compact.

Proof: Let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of $f(E)$. Then $\{f^{-1}(G_\alpha)\}_{\alpha \in A}$ is an open cover of E, by our previous theorem. Since E is compact, there's a finite set $B \subseteq A$ such that $\{f^{-1}(G_\alpha)\}_{\alpha \in B}$ covers E. But then $\{G_\alpha\}_{\alpha \in B}$ covers $f(E)$, so the cover $\{G_\alpha\}_{\alpha \in A}$ of $f(E)$ has a finite subcover. Thus $f(E)$ is compact.

Thus if:

$$\mathbb{R} \xrightarrow{\quad f \quad} \mathbb{R} \quad \text{is continuous}$$

$$\underset{\text{open}}{f^{-1}(U)} \longleftarrow\!\shortmid \; U \text{ open}$$

$$K \text{ compact} \;\shortmid\!\longrightarrow f(K) \text{ compact}$$

and since complements of closed sets are open:

$$\underset{\text{closed.}}{f^{-1}(C)} \longleftarrow\!\shortmid \; C \text{ closed}$$

Aside from continuous functions being uniformly continuous on compact sets, there's another significant property:

Corollary: If $f: E \longrightarrow \mathbb{R}$ is continuous and E is compact, then there exist $x_1, x_2 \in E$ such that for all $x \in E$

$$f(x_1) \leq f(x) \leq f(x_2).$$

ie. f attains an absolute maximum and an absolute minimum on E.

Proof. Since E is compact, $f(E)$ is compact. Set

$$y_1 = \inf f(E) \quad \text{and} \quad y_2 = \sup f(E).$$

Considering y_1: There are two possibilities, either $y_1 \in f(E)$ or y_1 is an accumulation point of $f(E)$. But if y_1 is an accumulation point of $f(E)$, then since $f(E)$ is closed by the Heine-Borel theorem, we must have $y_1 \in f(E)$. So in either case, $y_1 \in f(E)$. Similarly $y_2 \in f(E)$.

Therefore we can choose $x_1, x_2 \in E$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. Then by definition of y_1 and y_2, $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in E$.

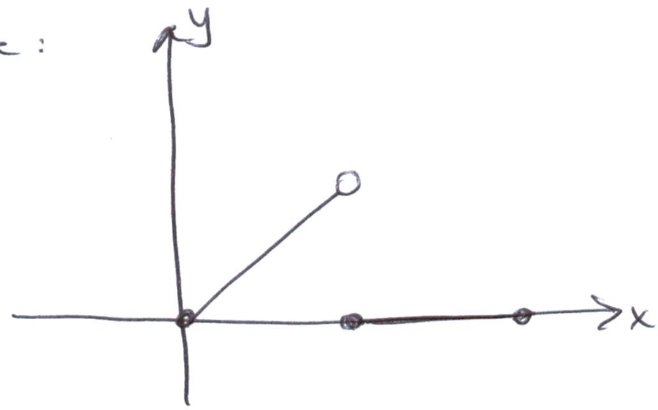
A couple of examples to illustrate the necessity of continuity and compactness:

Example: Consider $f: (0,1) \longrightarrow \mathbb{R}$ given by $f(x) = x$. Then f does not attain any maximum or minimum on its domain, because the image of $f(x)$ is the set $(0,1)$ which has no max or min.

<u>Example</u>: If $f:[0,2] \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

Then f looks like:



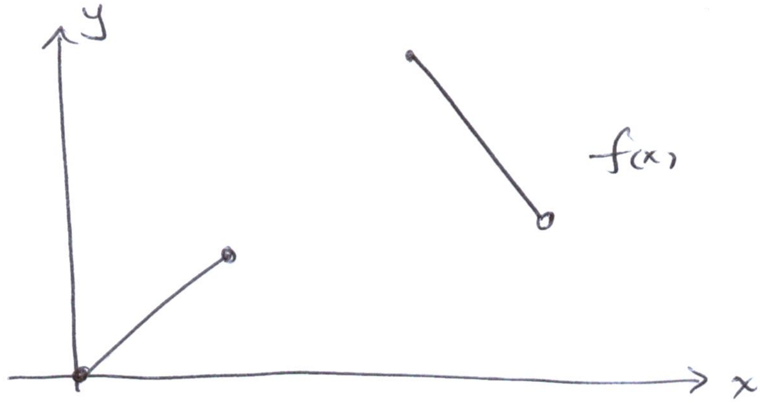
And again f does not achieve a max on its domain, because the image of f is $[0,1)$ and so has no maximum.

Compactness also interacts with continuity in the following way:

Suppose $f: E \longrightarrow \mathbb{R}$ is a one-to-one function. Then there's a function $f^{-1}: f(E) \longrightarrow E$. If f is continuous, then f^{-1} does not need to be continuous. For example, if $E = [0,1] \cup [2,3)$ we can set

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 4-x & \text{if } 2 \le x < 3 \end{cases}$$

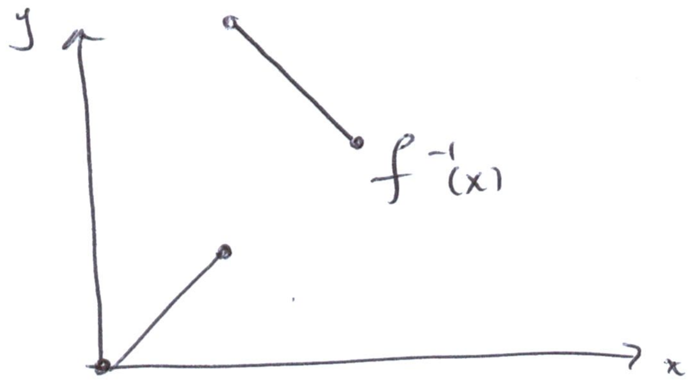
Then it is not too hard to check that f is 1-1 and continuous. It looks like:



One can check that $f(E) = [0,2]$ and that f has an inverse defined by

$$f^{-1}(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 4-x & \text{if } 1 < x \le 2, \end{cases}$$

which is <u>not continuous</u>.



The problem is that $E = [0,1] \cup [2,3)$ is not <u>compact</u>. If E were compact, then we have the following:

<u>Theorem</u>: Suppose $f: E \to \mathbb{R}$ is continuous and one-to-one, and that E is compact. Then $f^{-1}: f(E) \to E$ is continuous.

Proof: Let $\{y_n\}_{n=1}^{\infty}$ be any sequence in $f(E)$ converging to $y_0 \in f(E)$. We'll show that $\{f^{-1}(y_n)\}$ converges to $f^{-1}(y_0)$, which implies continuity of $f^{-1}: f(E) \to E$. To make notation easier, write x_n in place of $f^{-1}(y_n)$, so that $f(x_n) = y_n$.

Since E is compact, by Heine-Borel it's bounded and so $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Suppose that $\{x_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$, call its limit z_0. Then $z_0 \in E$ since E is closed, then by continuity of f the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$. But

$$\{f(x_n)\}_{n=1}^{\infty} = \{f(f^{-1}(y_n))\}_{n=1}^{\infty} = \{y_n\}_{n=1}^{\infty},$$

so it also converges to y_0. If x_0 is any point with $f(x_0) = y_0$, this gives $f(x_0) = f(z_0) \Rightarrow x_0 = z_0$. So then $\{x_n\}_{n=1}^{\infty}$ converges to x_0, i.e.

$$\{f^{-1}(y_n)\}_{n=1}^{\infty} \text{ converges to } f^{-1}(y_0).$$

So f^{-1} is continuous.