MATH 2080.

Added material: Continuity of multivariable functions

Example: Show that $f:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. To do this we must show that given $\varepsilon > 0$ there's a $\delta > 0$ such that $\|(x,y) - (0,0)\| = \|(x,y)\| < \delta$ implies $|f(x,y) - f(0,0)|$

$$= |f(x,y)| < \varepsilon.$$

First if $(x,y) = (0,0)$ then obviously $|f(x,y)| = |f(0,0)| = 0 < \varepsilon$, so we only consider $(x,y) \neq (0,0)$.

Let $\varepsilon > 0$. Then we make a couple preliminary observations: First, note $x^2 \leq x^2+y^2$, so when $(x,y) \neq (0,0)$

$$\frac{x^2}{x^2+y^2} \leq \frac{x^2+y^2}{x^2+y^2} = 1.$$

So in particular

$$\left|\frac{x^2y^4}{x^2+y^2} - 0\right| = \left|\frac{x^2y^4}{x^2+y^2}\right| \leq |y^4|,$$

and clearly

$$|y|^4 \leq \left|\sqrt{y^2+x^2}\right|^4 = \|(x,y)\|^4$$

So, choose $\delta = \varepsilon^{1/4}$. Then if $\|(x,y)\| < \delta$ we calculate

$$|f(x,y) - 0| = \left|\frac{x^2y^4}{x^2+y^2}\right| \le |y|^4 \le \|(x,y)\|^4 < \delta^4 = \varepsilon.$$

So $f(x,y)$ is continuous at $(0,0)$.

Example: Consider the function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is not continuous at $(0,0)$. Show that, in fact, there's no way to define $f(0,0)$ that make f continuous there.

To do this, we can use a sequential argument: Find sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ both converging to 0 such that $\{f(x_n,y_n)\}_{n=1}^{\infty}$ does not converge to $f(0,0)=0$.

Set $x_n = 0$, $y_n = \frac{1}{n}$. Then both clearly converge to 0 and $(x_n,y_n) \neq (0,0)$ for all n. We calculate

$$f(x_n,y_n) = f(0,\tfrac{1}{n}) = \frac{0-(\frac{1}{n})^2}{0+(\frac{1}{n})^2} = -1$$ for all n, so $\{f(x_n,y_n)\}_{n=1}^{\infty}$ converges to -1. So

$f(x,y)$ is not continuous.

To see there's no way of defining ~~$f(x,y)$~~ $f(0,0)$ that will make f continuous there, consider the second choice of sequences

$$x_n = 1/n, \quad y_n = 0, \quad \text{both converge to } 0.$$

Yet for this choice,

$$f(x_n, y_n) = \frac{(\frac{1}{n})^2 + 0}{(\frac{1}{n})^2 + 0} = 1 \quad \text{for all } n.$$

Thus $\{f(x_n, y_n)\}_{n=1}^{\infty}$ converges to 1.

So our first choice of sequences suggests we should define $f(0,0) = -1$ to have continuity at $(0,0)$. Our second choice of sequences suggests $f(0,0) = 1$. Since we have two contradicting values, f can't be made continuous at $(0,0)$.

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Note that we can introduce limits into our discussion of functions $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ in the obvious way. In analogy with functions $f: \mathbb{R} \longrightarrow \mathbb{R}$, we can define:

<u>Definition</u>: Let $A \subseteq \mathbb{R}^2$ and let $\vec{c} = (c_1, c_2) \in \mathbb{R}^2$ be given. Then c is an accumulation point of A if, for every $\varepsilon > 0$ there exists $\vec{x} = (x_1, x_2) \in A$

with $\|\vec{x}-\vec{c}\| < \varepsilon$ and $\vec{x} \neq \vec{c}$.

Then as before, we can take limits:

<u>Definition</u>: Let $A \subseteq \mathbb{R}^2$ and $f: A \to \mathbb{R}$. Suppose that $\vec{c} \in \mathbb{R}^2$ is an accumulation point of A. Then $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = L$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\vec{x} \in A$ and $0 < \|\vec{x}-\vec{c}\| < \delta$ then $|f(\vec{x}) - L| < \varepsilon$.

<u>Remark</u>: This is the same as for functions $f: \mathbb{R} \to \mathbb{R}$, but replacing absolute value with $\|\cdot\|$ for all elements in the domain of f.

As with limits of functions of one variable:

<u>Theorem</u>: Suppose $A \subseteq \mathbb{R}^2$, $\vec{x}_0 \in A$ and $f: A \to \mathbb{R}$. Then f is continuous at $\vec{x}_0$ if and only if $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$.

<u>Example</u>: Show that if $f: \mathbb{R}^2 \to \mathbb{R}$ has formula

$$f(x,y) = x^2 + 2y^2 - 3x + 4y + 5$$

then $\lim_{(x,y)\to(0,0)} f(x,y) = 5$.

To see this, let $\varepsilon > 0$.

Oberve that

$$|f(x,y)-5| = |(x^2+2y^2-3x+4y+5)-5|$$
$$= |x^2+2y^2-3x+4y|$$
$$\leq |x||x-3| + 2|y||y+2|.$$

If $\|(x,y)\| \leq 1$ then

$$|x-3| \leq |x|+3 \leq 1+3 = 4$$

and $|y+2| \leq |y|+2 \leq 1+3 = 3$

So if $\|(x,y)\| \leq 1$ then

$$|x||x-3| + 2|y||y+2|$$
$$\leq 4|x| + 6|y| < 4(1)+6(1) = 10.$$

Set $\delta = \min\{1, \frac{\varepsilon}{10}\}$. Then

$$|f(x,y)-5| \leq |x||x-3| + 2|y||y+2|$$
$$< |x|\cdot 4 + 6|y|$$
$$< \frac{4\varepsilon}{10} + \frac{6\varepsilon}{10} = \varepsilon.$$

So the stated limit holds.

MATH 2080

Section 3.2 Algebra of continuous functions.

Theorem: Suppose $f, g : D \to \mathbb{R}$ are continuous at $x_0 \in D$. Then

(i) $f+g$ is continuous at x_0

(ii) fg is continuous at x_0

(iii) If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

Proof: We can "cheat" our way to proofs of all these facts as follows:

If x_0 is not an accumulation point of D, then $f+g$, $\frac{f}{g}$ and fg are all continuous at x_0 "by default". If x_0 is an accumulation point of D, then continuity of f (or g) is equivalent to $\lim_{x \to x_0} f(x) = f(x_0)$ (resp. $\lim_{x \to x_0} g(x) = g(x_0)$). So, by our limit theorem (since we have the above equivalence)

$$\lim_{x \to x_0} (f+g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = f(x_0) + g(x_0) = (f+g)(x_0)$$

$$\lim_{x \to x_0} (fg)(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right) = f(x_0)g(x_0) = (fg)(x_0)$$

and so (i) and (ii) hold.

For (iii), when we write $\left(\frac{f}{g}\right)(x)$ we mean the function $\frac{f}{g} : D \to \mathbb{R}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, So by assumption $g(x) \neq 0$ for all $x \in D$. Also this means $\lim_{x \to x_0} g(x) \neq 0$ since $\lim_{x \to x_0} g(x) = g(x_0)$.

Thus
$$\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{f(x_0)}{g(x_0)} = \left(\frac{f}{g}\right)(x_0).$$

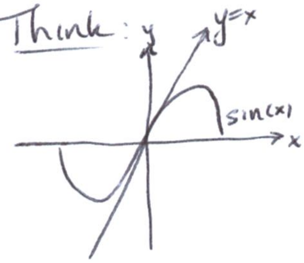
So the theorem holds.

<u>Remark</u>: Any one of these claims can also be proved directly from the definition. The book contains a proof of (iii) from the definition.

<u>Example</u>: All polynomials are continuous from this theorem, as are all rational functions $\frac{p(x)}{q(x)}$ wherever $q(x) \neq 0$.

<u>Example</u>: We can prove that the function $f(x) = \sin(x)$ is continuous everywhere by using only a couple basic facts about $\sin(x)$:

Think:



① For $0 \leq x \leq \pi/2$, $0 \leq \sin(x) \leq x$

② $\sin(-x) = -\sin(x)$

Then from this, we know $\lim_{x \to 0} \sin(x) = 0$ since

$-|x| \leqslant \sin(x) \leqslant |x|$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. So we can do a fairly straightforward "squeeze theorem" type of proof.

Let x_0 be any real number and let $\varepsilon > 0$. There's a trig identity

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

that we'll use in a moment. Set $\delta = \min\{\varepsilon, \pi\}$. If $|x_0 - y| < \delta$, then

$$|\sin x_0 - \sin y| = \left|2\cos\left(\frac{x_0+y}{2}\right)\sin\left(\frac{x_0-y}{2}\right)\right|$$

$$\left(\text{But } \left|\cos\left(\frac{x_0+y}{2}\right)\right| \leq 1 \text{ and } \underbrace{\left|\sin\left(\frac{x_0-y}{2}\right)\right| \leq \left|\frac{x_0-y}{2}\right|}_{\text{requires } |x_0-y|<\pi}\right)$$

$$\leq 2\left|\frac{x_0-y}{2}\right| = |x_0-y| < \delta = \varepsilon$$

So $\sin(x)$ is continuous at x_0. Next, since $\cos(x) = \sin(\frac{\pi}{2} - x)$, cosine is continuous at all x_0. Last, recall that cosine is zero on the set $\{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$, so that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ is continuous everywhere except on this set, by our previous theorem.

As a final result concerning functions and continuity, we consider the compositions of functions.

Theorem: If $f: D \to \mathbb{R}$ and $g: D' \to \mathbb{R}$ with $f(D) \subset D'$, and f is continuous at x_0 while g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

<u>Proof</u>: Let $\varepsilon > 0$.

There's $\delta_1 > 0$ such that if $|y - f(x_0)| < \delta_1$ and $y \in D'$ then $|g(y) - (g \circ f)(x_0)| < \varepsilon$, and there's δ_2 such that if $|x - x_0| < \delta_2$ and $x \in D$ then $|f(x) - f(x_0)| < \underline{\underline{\delta_1}}$ not ε!

Now we put the implications together:

If $|x - x_0| < \delta_2$ (with $x \in D$) then $|f(x) - f(x_0)| < \delta_1$ (and $f(x) \in D'$), and

so $|g(f(x)) - g(f(x_0))| < \varepsilon$. So $g \circ f$ is continuous at x_0.

Note that from this theorem and our relationship between continuous/limits, we can say that:

If f and g are as above, then

$\lim_{x \to x_0} f(x) = f(x_0)$, $\lim_{y \to f(x_0)} g(y) = g(f(x_0))$, and

therefore

$$(g \circ f)(x_0) = \lim_{x \to x_0} (g \circ f)(x) = g\left(\lim_{x \to x_0} f(x)\right),$$ so that

you can pass the limit inside of continuous functions (as taught in earlicar calc courses).

Thus, e.g.

$$\lim_{x \to \frac{\pi}{2}} \sqrt{\sin(x)} = \sqrt{\sin\left(\lim_{x \to \frac{\pi}{2}} x\right)}$$

$$= \sqrt{\sin(\tfrac{\pi}{2})} = 1.$$

MATH 2080

§ 3.3 Uniform continuity, compact, open and closed sets.

Some functions behave quite nicely with respect to continuity, in the following sense:

Suppose you want to show f is continuous at x_0. You proceed: Given $\varepsilon > 0$, there's a $\delta > 0$ that satisfies the definition at x_0. Now if you consider continuity at another point, y_0, and are given the <u>same</u> $\varepsilon > 0$, you do another calculation to find <u>another δ</u> satisfying the definition at y_0.

<u>I.e.</u> In general, the δ you find depends on ε <u>and</u> x_0.

For some functions, however, δ depends only on ε. These functions are called uniformly continuous.

<u>Definition</u>: A function $f: D \to \mathbb{R}$ is called uniformly continuous on $E \subset D$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. If f is uniformly continuous on D, we say f is uniformly continuous.

<u>Example</u>: Let $f: [\frac{5}{2}, 3] \longrightarrow \mathbb{R}$ be given by $f(x) = \dfrac{3}{x-2}$.

Let $\varepsilon > 0$.

Then

$$f(x)-f(y) = \frac{3}{x-2} - \frac{3}{y-2} = \frac{3(y-x)}{(x-2)(y-2)} = y-x \cdot 3 \cdot \frac{1}{(x-2)(y-2)}$$

The biggest $\frac{1}{(x-2)(y-2)}$ can get occurs when the product $(x-2)(y-2)$ is smallest, ie. when $x=y=\frac{5}{2}$.

Then $\frac{1}{(x-2)(y-2)} \leq \frac{1}{(\frac{5}{2}-2)(\frac{5}{2}-2)} = \frac{1}{(\frac{1}{2})^2} = 4.$

So since

$$|f(x)-f(y)| \leq |y-x| \cdot 3 \cdot \frac{1}{(x-2)(y-2)} \leq |y-x| \cdot 12,$$

choose $\delta = \frac{\varepsilon}{12}$. Then $|f(x)-f(y)| \leq 12|y-x| < 12 \cdot \frac{\varepsilon}{12} = \varepsilon$ whenever $|y-x| < \delta$. So f is uniformly continuous on $[5/2, 3]$.

<u>Example</u>: Consider $f(x) = 1/x$ on $(0,1)$. This is <u>not</u> uniformly continuous. To see this, we need only exhibit a single choice of ε for which no corresponding δ exists.

Choose $\varepsilon = 1$. Suppose $\delta > 0$. We want to find ~~the~~ $x, y \in (0,1)$ such that

- $\left|\frac{1}{x} - \frac{1}{y}\right| \geq \varepsilon = 1$, and
- $|x-y| < \delta$.

Choose $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Then

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{1/n} - \frac{1}{1/n+1}\right| = 1$$; so the first condition is satisfied for any choice of n. On the other hand, by choosing n large we can always guarantee that

$$|x-y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n(n+1)}\right| < \delta$$ for any choice of δ. Thus no δ will satisfy the definition of <u>uniform</u> continuity if we choose $\varepsilon = 1$.

On way of dealing with functions like $1/x$, instead of using the definition as above, is to use the following theorem:

<u>Theorem</u>: Suppose $f: D \to \mathbb{R}$ is uniformly continuous. If x_0 is an accumulation point of D, then $\lim_{x \to x_0} f(x)$ exists.

<u>Proof</u>: We use sequences. We must show: Given any sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D \setminus \{x_0\}$ for all n, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. In fact, it suffices to show that for every such sequence $\{x_n\}_{n=1}^{\infty}$, $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.

To this end, let $\varepsilon > 0$. Since f is uniformly continuous, there's $\delta > 0$ such that $x, y \in D$ and $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Since $\{x_n\}_{n=1}^{\infty}$ converges to x_0, it's Cauchy, so there exists N such that $m, n \geq N$ implies $|x_n - x_m| < \delta$. But x_n and x_m are in $D \setminus \{x_0\}$, and so by uniform continuity $|f(x_m) - f(x_n)| < \varepsilon$ for all $n, m \geq N$. Thus $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy, and <u>so converges.</u>

Thus in our previous example $f: (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, it suffices to point out that since $\lim\limits_{x \to 0} \frac{1}{x}$ does <u>not exist</u>, f cannot be uniformly continuous on $(0,1)$.

<u>Example</u>: There are functions that are <u>not</u> uniformly continuous and yet have limits everywhere, so the previous theorem doesn't work on all non-uniformly-continuous functions.

Consider $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. We know $\lim\limits_{x \to x_0} g(x)$ exists for all x_0. However, let $\varepsilon > 0$, and consider <u>any</u> $\delta > 0$. Choose x and y very close together:

$$|x - y| = \delta/2 < \delta$$

yet very big:

$$|x + y| = 3\varepsilon/\delta.$$

Then this pair x, y satisfies $|x-y| < \delta$, yet

$$|g(x) - g(y)| = \underset{\left(|x^2 - y^2|\right)}{\overset{\shortparallel}{|x+y||x-y|}} = \frac{3\varepsilon}{\delta} \cdot \frac{\delta}{2} = \frac{3}{2}\varepsilon > \varepsilon,$$

so $g(x)$ is <u>not</u> uniformly continuous.

<u>Definition</u>: A set $A \subseteq \mathbb{R}$ is <u>open</u> if and only if for ~~all~~ each $x \in A$ there's a nbhd Q of x with $Q \subseteq A$.

<u>Definition</u>: A set $E \subseteq \mathbb{R}$ is closed if and only if E contains all of its accumulation points.