Test 1

50 minutes

Please provide your answers in an answer booklet, this page will not be marked. The respective weight of each question is indicated next to it. Notes, calculators or cellphones are not allowed. For a proof to receive full credit it must be logically and mathematically sound, and every step must be properly justified.

Q1] State the following.

(a) [2 points] The definition of the infimum of a set $S \subset \mathbb{R}$.

Solution: An element $x \in \mathbb{R}$ is the infimum of S if it is a lower bound for S and if, given any other lower bound y of S, we have $y \leq x$.

(b) [2 points] What it means for a sequence $\{a_n\}_{n=1}^{\infty}$ to converge to a number A.

Solution: The sequence $\{a_n\}_{n=1}^{\infty}$ converges to A if for every $\epsilon > 0$ there exists N such that $|a_n - A| < \epsilon$ for all $n \geq N$.

(c) [3 points] What it means for a set $S \subset \mathbb{R}$ to be bounded above and below.

Solution: A subset S of $\mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. A subset S of $\mathbb{R}$ is bounded below if there exists $P \in \mathbb{R}$ such that $s \geq P$ for all $s \in S$

(d) [2 points] The Archimedean property for the real numbers $\mathbb{R}$.

Solution: There are a few equivalent ways of stating this property, depending on where you studied from. The following three answers were accepted: (1) The Archimedean property states that for all $x, y \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $nx > y$. (2) The Archimedean property states that for all $x, y \in \mathbb{R}$ with $x, y > 0$ there exists $n \in \mathbb{N}$ such that $nx > y$. (3) There Archimedean property states that for all $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > x$.

(e) [2 points] The triangle inequality.

Solution: $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

(f) [2 points] The least upper bound property of the real numbers.

Solution: Every nonempty subset of real numbers that is bounded above has a least upper bound.

Q2]. . . [6 points] Prove that

$$\left\{\frac{1}{\sqrt{n}+1}\right\}_{n=1}^{\infty}$$

converges to zero.

Solution: Let $\epsilon > 0$ be given.

Then we have $\left|\frac{1}{\sqrt{n}+1} - 0\right| < \epsilon$ if and only if $\frac{1}{\sqrt{n}+1} < \epsilon$, since the quantity in the absolute value is always positive. This inequality holds if and only if $\sqrt{n}+1 > \frac{1}{\epsilon}$, which holds if and only if $n > (\frac{1}{\epsilon} - 1)^2$. Therefore choosing $N > (\frac{1}{\epsilon} - 1)^2$ guarantees $\left|\frac{1}{\sqrt{n}+1} - 0\right| < \epsilon$ for all $n \geq N$, meaning the sequence converges to 0.

Q3]. . . [6 points] Let $S \subset \mathbb{R}$ be nonempty and bounded from above. Set $T = \{t \mid t = -s \text{ for some } s \in S\}$. Show that if $B = \sup S$ then $-B = \inf T$.

Solution: First, since $B = \sup S$ we have $s \leq B$ for all $s \in S$, meaning $-B \leq -s = t$ for all $t \in T$. Thus $-B$ is a lower bound for T. Now, if C is any other lower bound for T, then $C \leq t$ for all $t \in T$ yields $C \geq -t = s$ for all $s \in S$, so that $-C$ is an upper bound of S. Since B is the supremum of S, this gives $B \leq -C$, and thus $-B \geq C$. This proves $-B$ is the infimum of T.

Q4]. . . [5 points] Show that the set

$$X = \left\{\frac{a}{b} + \sqrt{2} \;\middle|\; a, b \in \mathbb{N} \text{ and } a, b \text{ are both odd}\right\}$$

is countable. Use any theorems from class that you like, but say which theorems you are using when they are used.

Solution: Define a function $f : X \to \mathbb{Q}$ by $f(x) = x - \sqrt{2}$. Then f is one-to-one since $f(\frac{a}{b} + \sqrt{2}) = f(\frac{c}{d} + \sqrt{2})$ implies $\frac{a}{b} = \frac{c}{d}$ and thus $\frac{a}{b} + \sqrt{2} = \frac{c}{d} + \sqrt{2}$. Now observe that since $\mathbb{Q}$ is countable (proved in class), so is the subset $f(X)$ (since a subset of a countable set is countable, proved in class). Since f provides a one-to-one, and onto function from X to $f(X)$, X is countable.