Name: ______________________________

Student Number: ______________________________

I understand that cheating is a serious offence:

Signature: ______________________________

(*In Ink*)

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 26 pages including this cover page, the backs of pages and blanks pages for Question 6. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 70 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question, or clearly indicate that your solution continues on the blank pages at the end of the booklet. Unjustified answers will receive little or no credit.

V. Please do not call or e-mail your instructor to inquire about grades. They will be available shortly after they have been marked.

For rough work

1. State the following:

[2] (a) The well ordering principle.

[2] (b) The definition of a dense subset $S \subset \mathbb{R}$.

[2] (c) The least upper bound property of the real numbers.

[2] (d) What it means for a sequence $\{a_n\}_{n=1}^{\infty}$ to converge to a number L.

[2] (e) The definition of a neighbourhood of $x \subset \mathbb{R}$.

For rough work

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[2] (f) The definition of the supremum of a set $S \subset \mathbb{R}$.

[2] (g) The definition of a Cauchy sequence.

[2] (h) The Bolzano-Weierstrass Theorem.

[2] (i) The definition of an increasing sequence and the definition of a decreasing sequence.

[2] (j) What it means for a function $f : D \to \mathbb{R}$ to have a limit L at an accumulation point x_0 of D.

For rough work

[2] (k) What it means for a function $f : D \to \mathbb{R}$ to be continuous at a point $x_0 \in D$.

[2] (l) The Heine-Borel Theorem.

[2] (m) The definition of a closed set.

[2] (n) The definition of a compact set.

[2] (o) What it means for a function to be uniformly continuous.

For rough work

[5] 2. Use the definition of convergence to show that $\{\frac{2n+1}{2n-5}\}_{n=1}^{\infty}$ converges to 1.

For rough work

[5] 3. Use the definition of uniform continuity to show that $f(x) = x^2$ is uniformly continuous on $[-4, 6]$.

For rough work

[5] 4. Use the definition of the limit of a function $f : \mathbb{R}^2 \to \mathbb{R}$ to show that if $f(x, y) = x^2 + y - 1$ then

$$\lim_{(x,y)\to(0,0)} f(x, y) = -1.$$

For rough work

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[5] 5. Prove directly from the definition of compactness (i.e. do not use the Heine-Borel theorem) that $(-1, 1)$ is not compact.

For rough work

[20] 6. Do any TWO of the following problems, each is worth 10 points. Write your solutions on the blank pages that follow. If you attempt more than two problems, circle the two problems below which you want to be graded. If you attempt more than two problems and do not circle the ones you want graded, the first two appearing in the subsequent pages will be marked. If you use any theorems from class, clearly indicate where in your proofs they are used. Unjustified steps in your proofs will receive little or no credit.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and has the property that for every $\epsilon > 0$ there is $M > 0$ such that if $|x| \geq M$ then $|f(x)| \leq \epsilon$. Show that f is uniformly continuous.

2. Prove that a set $E \subset \mathbb{R}$ is closed if it contains all limits of sequences of members of E. That is, prove that E is closed if for every convergent sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in E$ for all n, the limit of the sequence is contained in E.

3. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences converging to A and B respectively, prove that $\{a_n b_n\}_{n=1}^{\infty}$ converges to AB. This can be done in two steps:
(a) Prove that every convergent sequence is bounded.
(b) Use boundedness and the equality

$$|a_n b_n - AB| = |a_n(b_n - B) + B(a_n - A)|$$

to show that $\{a_n b_n\}_{n=1}^{\infty}$ converges to AB.

4. Let A and B be sets of real numbers. For each statement, either provide a proof that it is true or a counterexample showing it is false.
(a) Define $A - B = \{a - b \mid a \in A, b \in B\}$. Then

$$\sup A - \inf B = \sup(A - B).$$

(b) Define $AB = \{ab \mid a \in A, b \in B\}$. Then

$$\sup A \sup B = \sup(AB).$$

For rough work

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