MATH 1500

§4.7 Optimization.

Optimization problems are word problems, like related rates problems. However in an optimization problem, nothing is changing with time. Instead you typically must maximize or minimize some function.

Steps:

1. Draw a diagram, introduce notation (ie name the quantities involved
2. Find the function to be maximized, and the domain of the function (an interval)
3. Solve for the max/min on the interval. Be sure to use the first/second derivative test or some other reasoning to justify your answer, if necessary.

Example: A farmer has 2400 m of fencing, and wishes to enclose a rectangular field that borders a river. He needs no fence along the river. What is the largest area he can enclose?

Solution: Draw a picture; name variables:

(1)

Then we know Area $= xy$, while $2x + y = 2400$ since we want to use all the fence.

(2) The function to be maximized is Area $= xy$, but it's a function of two variables. We only know how to maximize functions of 1 variable, so use $2x + y = 2400 \Rightarrow y = 2400 - 2x$.

Then Area $= A(x) = x(2400 - 2x) = 2400x - 2x^2$

Where is the interval in this problem? It comes from real-world considerations:

- $x \geq 0$ since x is the length of a fence, so positive.
- $x \leq 1200$ since making $x = 1200$ uses up all the fencing, so we can't go any larger.

Interval is $[0, 1200]$.

(3) Maximize $A(x) = 2400x - 2x^2$ on $[0, 1200]$.

Critical points:

$$A'(x) = 2400 - 4x$$

So $2400 - 4x = 0 \Rightarrow x = 600$.

We plug in critical points, and endpoints. The largest will be the absolute max.

$A(0) = 2400(0) - 2(0) = 0$

$A(1200) = 0$

$A(600) = 2400(600) - 2 \cdot 600^2 = 1\,440\,000 - 720\,000 = 720\,000\,m^2$.

There is no need for a derivative test, as this is an absolute max/min over an interval.

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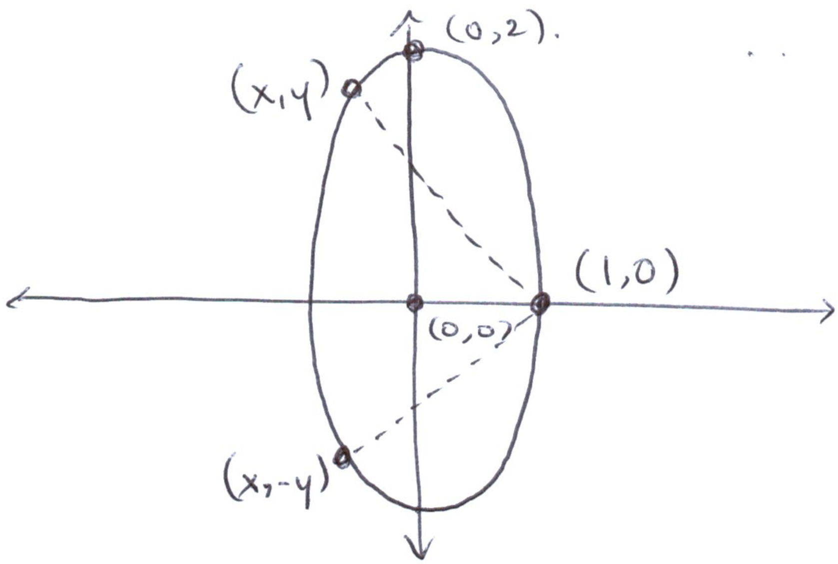
(19) Find the point (x,y) on the ellipse $4x^2 + y^2 = 4$ that is farthest away from the point $(1,0)$.

Solution: The ellipse is

(1)



From the symmetry of the picture it seems likely that we should find two points that maximize the distance!

(2) We want to maximize the distance

$$d(x,y) = \sqrt{(x-1)^2+(y-0)^2}$$
$$= \sqrt{(x-1)^2+y^2}.$$

Trick to make the formula nicer: We want $d(x,y)$ to be as big as possible. Wherever this happens, $(d(x,y))^2 = (x-1)^2+y^2$ will be maximized, too. So we make use of this and only maximize

$$(d(x,y))^2 = (x-1)^2+y^2$$ to keep things simpler.

Call this function S. We make S a function of one variable by substituting $y^2 = 4-4x^2$:

$$S(x) = (x-1)^2+(4-4x^2) = 5-2x-3x^2.$$

The x-coordinates of points on the ellipse are between -1 and 1, so we maximize on $[-1,1]$.

(3) Maximize:

$$S'(x) = -2-6x, \quad \text{so} \quad S'(x)=0 \Rightarrow x = -\frac{1}{3}.$$

We test $-1, -1/3, 1$ and see which gives the greatest value for $S(x)$:

$$S(-1) = 5-2(-1)-3(-1)^2 = 5+2-3 = 4$$

$$S(1) = 5-2(1)-3(1)^2 = 0.$$

$$S(\tfrac{1}{3}) = 5-2(\tfrac{-1}{3})-3(\tfrac{-1}{3})^2 = 5+\frac{2}{3}-\frac{1}{3} = \frac{16}{3}.$$

Since $\frac{16}{3}$ is the largest, the max occurs there.
Thus we plug $x = -\frac{1}{3}$ to find the y-coordinate of the most distant point(s):

$$4\left(-\tfrac{1}{3}\right)^2 + y^2 = 4$$

$$\Rightarrow \frac{4}{9} + y^2 = 4$$

$$\Rightarrow y = \pm\sqrt{\frac{32}{9}}.$$

Example: Two numbers sum to give 42. What is the largest their product could be?

Solution: ① No picture here! Call the numbers x and y.

② We want $x+y=42$ and xy as big as possible.

$$\text{Product} = p(x) = xy = x(42-x)$$
$$= 42x - x^2$$

The interval is $[0, 42]$, since x can be any of these numbers.

③ $p'(x) = 42 - 2x$

So $p'(x) = 0 \Rightarrow 42 - 2x = 0 \Rightarrow x = 21$.

Therefore $y = 21$ as well, and this x and y give the max product.

Remark: Observe that this problem, while worded differently, contains almost all the same elements as the "maximize fenced area" problem. Indeed, you can think of this problem as 42m of fence and maximize the area enclosed (if rectangular).

Lesson: Do many practice problems of this kind. Having done enough, you'll have seen most "templates", even if the story that's told to go along with the template is different.