MATH 1230

§2.6 Higher-order derivatives.

After differentiating a function once to get $f'(x)$, we can differentiate $f'(x)$ to get $f''(x)$. There are many notations for this, as before. If $y=f(x)$ then

$$f''(x) = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x).$$

We can differentiate many times, say n times, and get

$$f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}f(x).$$

We usually write primes (y', y'', y''') up to the third derivative, and then after that we write $y^{(4)}, y^{(5)}, \dots, y^{(n)}$.

Example: If an object is a distance $d(t)$ from the ground at time t, then $d'(t)$ is the rate of change of the distance: it is therefore the velocity:

$$v(t) = d'(t).$$

The acceleration of the object is then the rate of change of its velocity, so

$$a(t) = v'(t) = (d'(t))' = d''(t).$$

Remark: The third derivative, $d'''(t)$, is also occasionally called the "jerk".

<u>Example</u>: Sometimes the higher derivatives of a function exhibit an obvious pattern. E.g. if $y = \sin(x)$, then

$$y' = \cos(x), \quad y'' = -\sin(x), \quad y''' = -\cos(x), \quad y^{(4)} = \sin(x) \ldots$$

and then it repeats. So in general:

$$y^{(n)} = \begin{cases} \sin(x) & \text{if } n \text{ is divisible by } 4 \text{ (ie } n = 4k) \\ \cos(x) & \text{if } n = 4k+1 \\ -\sin(x) & \text{if } n = 4k+2 \\ -\cos(x) & \text{if } n = 4k+3 \end{cases} \quad \text{for some } k.$$

Other times the pattern is less obvious and requires a bit of work to formally verify.

<u>Example</u>: If $f(x) = \frac{1}{x}$, what is a formula for $f^{(n)}(x)$? Prove your formula works.

<u>Solution</u>: Let's test a few cases to get an idea:

$$f'(x) = \frac{-1}{x^2} = -x^{-2}$$

$$f''(x) = (-1)(-2) = x^{-3}$$

$$f'''(x) = (-1)(-2)(-3)x^{-4}$$

$$f^{(4)}(x) = (-1)(-2)(-3)(-4)x^{-5} \ldots \text{ etc.}$$

So the pattern becomes clear. Let's prove that our guess of

$$f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$$

works.

Proof of our formula: For things that are easily seen to be true for small integers (but harder to see for bigger integers) we use induction to prove them.

Step 1: Check the "base case". Here that means that we verify the formula $f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$ holds for $n=1$. So if $n=1$, direct calculation gives $f'(x) = \frac{-1}{x^2} = (-1)\, x^{-2}$, whereas the formula gives

$f^{(1)}(x) = (-1)^1\, 1!\, x^{-1-1} = -x^{-2}$, so it works.

Step 2: Verify that if the formula is true for some value k, then it's true for $k+1$. (The "induction step").

So suppose $y^{(k)} = (-1)^k k!\, x^{-k-1}$; then differentiating:

$$f^{(k+1)}(x) = \frac{df^{(k)}}{dx}(x) = \frac{d}{dx}\left[(-1)^k k!\, x^{-k-1}\right]$$

$$= (-1)^k k! \frac{d}{dx} x^{-k-1}$$

$$= (-1)^k k!\, (-k-1)\, x^{-k-1-1}$$

$$= (-1)^{k+1} k!\, (k+1)\, x^{-(k+1)-1}$$

$$= (-1)^{k+1} (k+1)!\, x^{-(k+1)-1}$$

So the formula is true for $k+1$. We conclude that the formula holds for all n by induction.

For some higher derivatives, the pattern is much less clear.

<u>Example</u>: Calculate the higher derivatives of

$f(x) = \sqrt{x^2+1}$.

<u>Solution</u>: We'll do $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$. Each will involve the chain rules and product rule:

$$f(x) = \sqrt{x^2+1} = (x^2+1)^{1/2}$$

$$f'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \boxed{= (x^2+1)^{-1/2} \cdot x}$$

$$f''(x) = \frac{1}{2}\left[\left((x^2+1)^{-1/2}\right)' \cdot 2x + (x^2+1)^{-1/2} \cdot (2x)'\right]$$

$$= \frac{1}{2} \cdot \frac{-1}{2}(x^2+1)^{-3/2} \cdot 2x \cdot 2x + (x^2+1)^{-1/2} \cdot 2 \cdot \frac{1}{2}$$

$$= -(x^2+1)^{-3/2} \cdot x^2 + (x^2+1)^{-1/2}$$

$$= (x^2+1)^{-3/2}(-x^2 + x^2+1) \boxed{= (x^2+1)^{-3/2}}$$

$$f'''(x) = -3/2\,(x^2+1)^{-5/2} \cdot 2x = (-3x)(x^2+1)^{-5/2}.$$

$f^{(4)}(x)$ is a mess, but it comes out to be

$$f^{(4)}(x) = 3(4x^2-1)(x^2+1)^{-7/2}.$$

<u>Remark</u>: There <u>is</u> a formula for stuff like this, it's just intractable at times.

There are also rules for higher-order derivatives that we will not discuss much in this class:

Example: What is the n^{th} derivative of $f(x)g(x)$, assuming $f(x)$ and $g(x)$ have n derivatives (ie. they are n times differentiable).

Solution: Let's take a few derivatives to test:

$$(fg)' = f'g + g'f$$

$$(fg)'' = (f'g + g'f)' = f''g + f'g' + g''f + g'f'$$

$$= f''g + 2f'g' + g''f$$

$$(fg)''' = (f''g + 2f'g' + g''f)' = f'''g + f''g' + 2f''g' + 2f'g'' + g'''f + g''f'$$

$$= f'''g + 3f''g' + 3f'g'' + g'''f.$$

Compare this to:

$$a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a^2 + 2ab + b^2)(a+b) = a^3 + 2a^2b + b^2a + a^2b + 2ab^2 + b^3$$

$$= a^3 + 3a^2b + 3b^2a + b^3$$

and in general, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \text{(binomial theorem)}$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

"n choose k"

From the similarities in cases $n=1, 2, 3$ one might guess

$$(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

and this guess is correct (though an actual proof would require induction).

There are also higher-order rules for things like the chain rule:

$$\frac{d^n f(g(x))}{dx^n} = \text{———— stuff ————}$$

but the formulas are extremely messy and useless.

MATH 1230

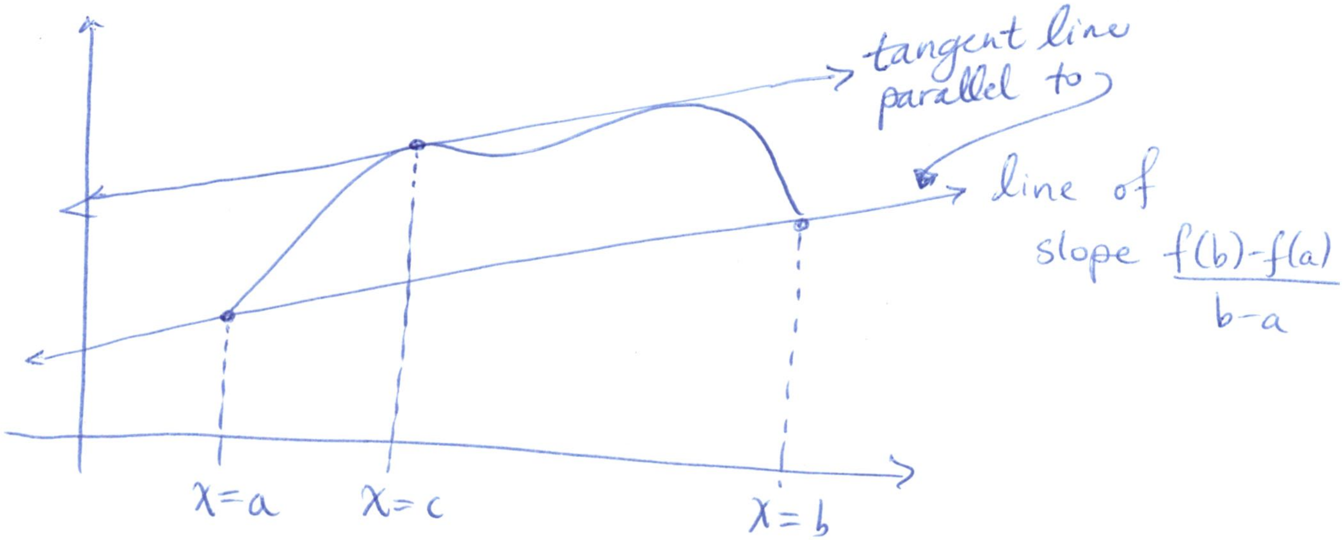
§2.8 The Mean Value Theorem

The Mean Value Theorem says the following:

Theorem: Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

In pictures, this says: The slope of the line connecting $(a, f(a))$ and $(b, f(b))$ must be equal to the slope of $f(x)$ at some point in (a,b).



This theorem will allow us to interpret derivatives in ways that provide information about the graph of a function — specifically, maxes, mins, and increasing/decreasing behaviour.

For example, here is what we can say about maxes/mins

<u>Theorem</u>: If $f(x)$ is defined on (a,b) and has a maximum (or minimum) at c in (a,b), and if $f'(c)$ exists, then $f'(c)=0$.

<u>Proof</u>: Consider the case when $f(c)$ is a max, the case of a min is similar.

Since $f(c)$ is a max, $f(x)-f(c) \le 0$ for all x in (a,b). So

$$\underbrace{\frac{f(x)-c}{x-c} \le 0}_{\text{when } c \text{ is left of } x} \implies \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le 0$$

$$\implies f'(c) \le 0.$$

and also

$$\underbrace{\frac{f(x)-c}{x-c} \ge 0}_{\text{when } c \text{ is right of } x} \implies \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$$

$$\implies f'(c) \ge 0.$$

Overall, $f'(c)=0$.

As in the case of the chain rule, the full proof is more involved than necessary for our purposes. So, we will consider a special case known as Rolle's Theorem:

Theorem (Rolle): Suppose $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $g(a)=g(b)$ then there is a point c with c in (a,b) and $g'(c)=0$.

Proof: If $g(x)=g(a)$ for all $x \in [a,b]$ then g is a constant function, so $g'(x)=0$.

If $g(x)$ is not a constant function, we can find x_0 in $[a,b]$ with $g(x_0) \neq g(a)$, suppose $g(x_0) > g(a)$. By the Min/Max Theorem, $g(x)$ must have a maximum at some point c in $[a,b]$, and so $g(c) \geq g(x_0) > g(a)$. Therefore c is not a or b, so c is in the interval (a,b) and thus $g'(c)$ exists. By the previous theorem, $g'(c)=0$.

In full generality, the proof appears on page 142, as well as a statement of the Generalized MVT, which we will not use.

To further connect the shape of the graph of $f(x)$ with the properties of $f'(x)$, we need some definitions.

Definition: Let $f(x)$ be defined on I containing x_1 and x_2.

(i) If $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ then f is increasing on I.

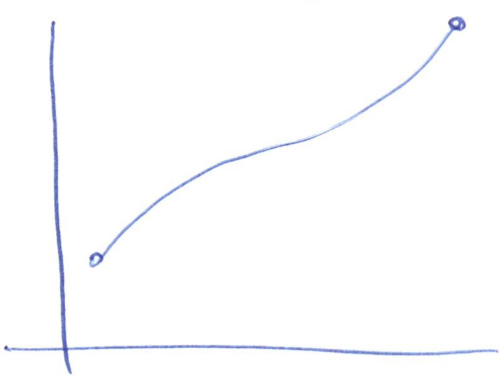
(ii) If $f(x_1) < f(x_2)$ whenever $x_1 > x_2$ then f is decreasing on I.

(iii) If $f(x_1) \leq f(x_2)$ whenever $x_1 > x_2$ then f is non increasing on I

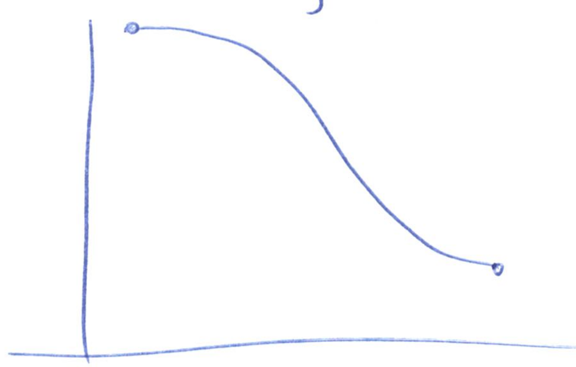
(iv) If $f(x)_1 \geq f(x_2)$ whenever $x_1 > x_2$ then f is non decreasing on I.

Each of these has a corresponding picture:

Increasing



Decreasing



Non-decreasing: same as above, but it's allowed to be flat in some places

Non-increasing: Same as above, but allowed to be flat in some places.

<u>Theorem</u>: Let $f(x)$ be a function defined on I. Then:

(i) If $f'(x) > 0$ for all x in I, then $f(x)$ is increasing on I.

(ii) If $f'(x) < 0$ for all x in I, then $f(x)$ is decreasing on I.

(iii) $f'(x) \geq 0 \Rightarrow$ nondecreasing

(iv) $f'(x) \leq 0 \Rightarrow$ nonincreasing.

<u>Proof</u>: Suppose x_1, x_2 are in I and $x_1 > x_2$. By MVT there is a point c with

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

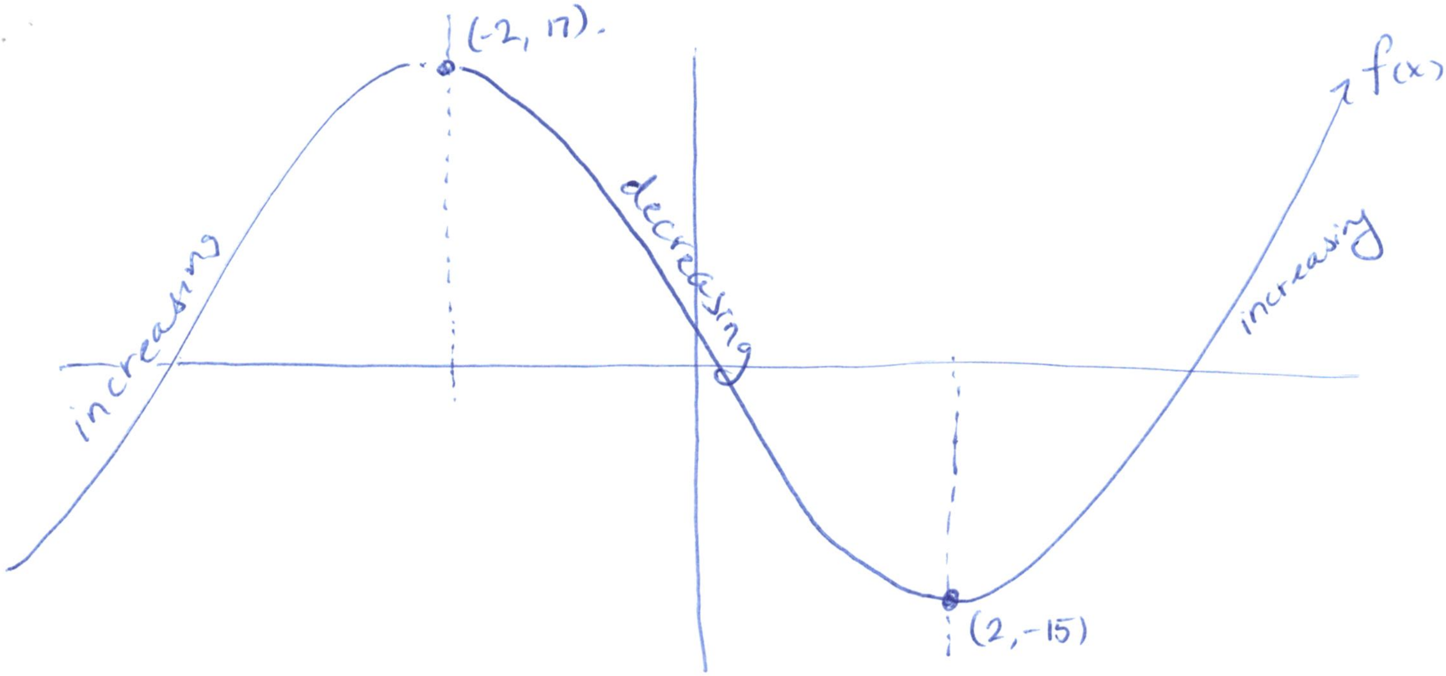
So $(x_1 - x_2) f'(c) = f(x_1) - f(x_2)$. Since $x_1 - x_2 > 0$, $f(x_1) - f(x_2)$ will have the same sign as $f'(c)$ and ~~may~~ will be zero whenever $f'(c)$ is zero.

<u>Example</u>: Consider $f(x) = x^3 - 12x + 1$. Then

$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$.

Then
- $f'(x) = 0$ if $x = 2$ or $x = -2$.
- $f'(x) < 0$ if $-2 < x < 2$
- $f'(x) > 0$ if $x < -2$ or $x > 2$.

Therefore we can make a rough sketch of $f(x)$:



Calculate $(2, f(2)) = (2, -15)$

$(-2, f(2)) = (-2, 17)$

MATH 1230

§2.9 Implicit differentiation.

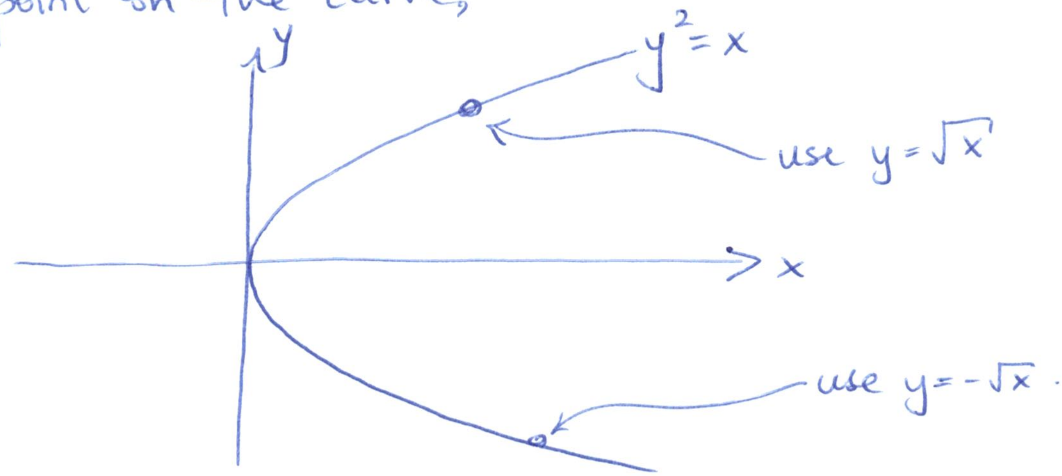
Implicit differentiation is a method of differentiating an equation when we cannot solve for y. For example, if $y^3 + xy^2 - \sin(y) = 1$, how could we ever compute $\frac{dy}{dx}$? Answer: Implicit differentiation.

The technique is to treat y as a function $y(x)$, applying chain and product rules as needed.

Example: Find $\frac{dy}{dx}$ if $y^2 = x$.

Solution: A naive approach ~~may~~ would be to do the following:

If $y^2 = x$, then $y = \pm\sqrt{x}$. So depending on the point on the curve,



we would use either $y_1 = \sqrt{x}$ or $y_2 = -\sqrt{x}$ to describe the curve near it.

Using traditional derivatives,

$$y_1' = \frac{1}{2\sqrt{x}} \quad \text{and} \quad y_2' = \frac{-1}{2\sqrt{x}}$$

together describe how to find slopes of tangent lines to the curve $y^2 = x$. But there IS an easier way!

Differentiate both sides of $y^2 = x$ with respect to x

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

$$\Rightarrow \underbrace{2y \cdot \frac{dy}{dx}}_{\text{chain rule.}} = 1 \Longrightarrow \boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

Observe: This answer works for <u>all</u> points on the curve $y^2 = x$ and agrees with y_1', y_2' above.

<u>Example</u>: Find the slope of the tangent to $x^2 + y^2 = 25$ at the point $(3, -4)$.

<u>Solution</u>: We simplify things by using implicit differentiation instead of breaking the equation of the circle into two functions.

Compute: $$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$

Thus the slope at $(3,-4)$ is $\frac{-3}{-4} = \frac{3}{4}$.

Remark: Sometimes the "think of y as a function of x, and differentiate" instruction is confusing/unclear. If it's not clear why $(y^3)' = 3y^2y'$, try replacing y with an actual function of x and see what you get:

E.g if $y = \sin(x)$ then

$$\underbrace{\left((\sin(x))^3\right)'}_{(y^3)'} = 3(\sin(x))^2 \cdot \cos(x)$$

$$(y^3)' = 3y^2 \cdot y'.$$

Hopefully this will prevent any mistakes, as it provides a "check" for your implicit differentiation reasoning.

Example Calculate $\frac{dy}{dx}$ if $y^3 + xy^2 - \sin(y) = 1$.

Solution: Differentiate both sides with respect to x.

$$\frac{d}{dx}\left(y^3 + xy^2 - \sin(y)\right) = \frac{d}{dx}(1)$$

$$\Rightarrow 3y^2 \frac{dy}{dx} + \underbrace{\left[\frac{d}{dx}(x) \cdot y^2 + x \cdot \frac{d}{dx} y^2\right]}_{\text{product rule}} - \cos(y)\frac{dy}{dx} = 0.$$

$$\Rightarrow 3y^2 \frac{dy}{dx} + y^2 + x \cdot 2y\frac{dy}{dx} - \cos(y)\frac{dy}{dx} = 0.$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(3y^2+2xy-\cos(y)\right) = -y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{3y^2+2xy-\cos(y)}.$$

Implicit differentiation can cause a few problems, however, so you need to be on the lookout for ~~such~~ issues. arising from your method.

For example, the equation that you start with may not define y as a function of x, so think of y as $y(x)$ loses meaning.

Example: Consider the function defined by

$$(x^2+y^2)^2 = x^2-y^2.$$

What is the slope of the tangent line at $(0,0)$?

Solution: Blindly applying implicit differentiation rules would give:

$$x^4+2x^2y^2+y^4 = x^2-y^2$$

$$\Rightarrow \frac{d}{dx}(x^4+2x^2y^2+y^4) = \frac{d}{dx}(x^2-y^2)$$

$$4x^3 + 2\left(2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3 = 2x - 2y\frac{dy}{dx}$$

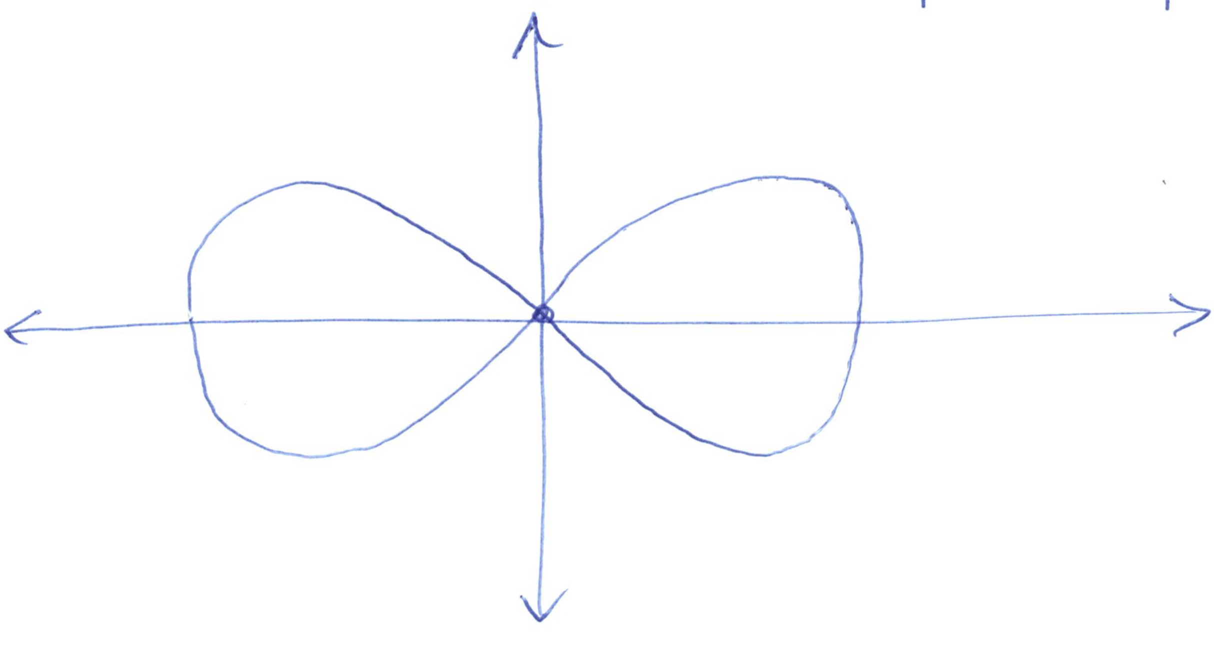
$$\Rightarrow \quad 4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3 = 2x - 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(4x^2y + 2y\right) = 2x - 4y^3 - 4x^3 - 4xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x - 4y^3 - 4x^3 - 4xy^2}{4x^2y + 2y}$$

So at $(0,0)$ we get $\frac{dy}{dx} = \frac{0}{0}$, undefined.

What went wrong here? Well, at $(0,0)$ y is <u>not</u> a function of x. The graph of $(x^2+y^2)^2 = x^2 - y^2$ is



and so at $(0,0)$ we have , which is not the graph of a function.

Moral of the story: Just because implicit differentiation allows you to symbolically find an equation containing $\frac{dy}{dx}$ does not mean the equation IS valid.

The validity of implicit differentiation at a point (x,y) is rooted in something called the Implicit Function Theorem, and is beyond this course.

As with "regular" derivatives, we can also calculate higher derivatives implicitly.

Example: If $xy+y^2=2x$, then

$$\frac{d}{dx}(xy+y^2)=\frac{d}{dx}2x$$

$$\Rightarrow y+x\frac{dy}{dx}+2y\frac{dy}{dx}=2$$

$$\Rightarrow \frac{dy}{dx}=\frac{2-y}{x+2y}$$

And
$$\frac{d}{dx}\left(y+x\frac{dy}{dx}+2y\frac{dy}{dx}\right)=\frac{d}{dx}2$$

$$\Rightarrow \frac{dy}{dx}+\frac{d^2y}{dx^2}x+\frac{dy}{dx}+2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}=0.$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{-2\frac{dy}{dx}-2\left(\frac{dy}{dx}\right)^2}{2y+x}$$

(now plug in $\frac{dy}{dx}=\frac{2-y}{x+2y}$)