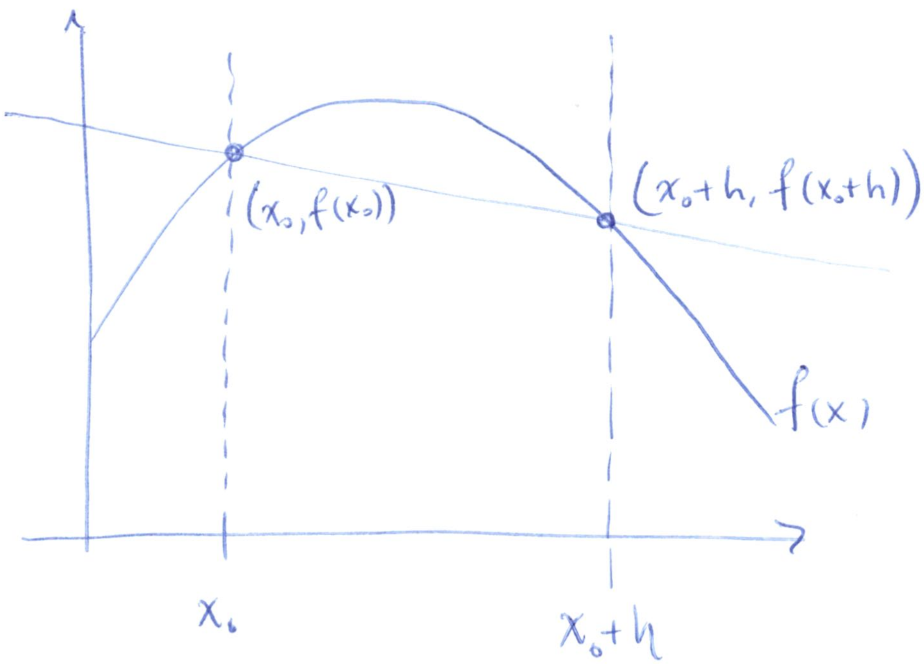
MATH 1230

§2.1 (Skip §9.1 as originally planned).

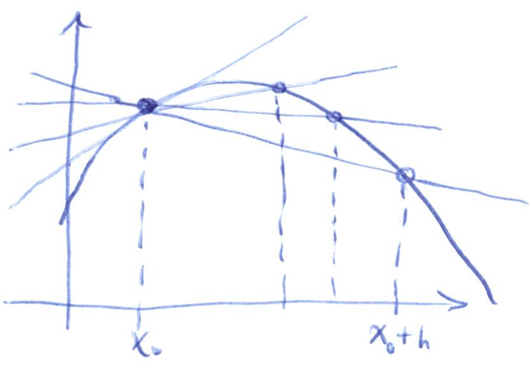
Let $f(x)$ be a continuous function, and suppose that x_0 is a point in the interior of the domain of $f(x)$.

Consider a line passing through two points x_0 and x_0+h $(h>0)$ on the graph of $f(x)$:



The slope of this line is $\dfrac{f(x_0+h)-f(x_0)}{h}$.

As the points x_0 and x_0+h move closer together, the slope of the line more closely approximates the "slope" of $f(x)$ at the point x_0.



The tangent line to the graph of $f(x)$ at the point $P = (x_0, f(x_0))$ is the line through P with slope

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}, \text{ provided the limit exists.}$$

So its equation is $y = m(x - x_0) + f(x_0)$.

If the limit does not exist, then there is no non-vertical tangent line at P.

If the limit does not exist but is $\pm\infty$, ie

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \pm\infty,$$

then we say the tangent line is vertical and its equation is $x = x_0$.

Example: Find the equation of the line tangent to $f(x) = x^2$ at $x_0 = 2$ (So $P = (2, 4)$).

Solution: Here

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} 4 + h = 4.$$

So the tangent line is $y = 4(x-2)+4$.

Example: Find the tangent line to $f(x) = \sqrt[3]{x}$ at $x_0 = 0$.

Solution: We compute

$$m = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty.$$

Thus the tangent line to $f(x) = \sqrt[3]{x}$ is vertical at $x = 0$, so its equation is $x = 0$.

Example: Show $f(x) = x^{2/3}$ has no tangent line at $x_0 = 0$.

Solution: Again, we want to consider $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$,

but this time we want to show it does not exist instead of evaluating it. So we consider left and right limits:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{2/3}}{h} = \lim_{h \to 0^-} \frac{1}{h^{1/3}} = -\infty,$$

since $h^{1/3} < 0$ when $h < 0$.

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{1}{h^{1/3}} = +\infty,$$

since $h^{1/3} > 0$ when $h > 0$.

The slope of the line tangent to $f(x)$ at $x = x_0$ is also sometimes called "the slope of $f(x)$ at $x = x_0$".

We can also describe the slope of $f(x)$ at the endpoints of its domain by using left/right limits.

If the domain is $[a, b]$, then

$$\text{slope at } x = a = \lim_{x \to a^+} \frac{f(a+h) - f(a)}{h}$$

$$\text{slope at } b = \lim_{x \to b^-} \frac{f(b+h) - f(b)}{h}.$$

Definition: A function $f(x)$ is differentiable at a point x_0 in the interior of its domain if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \underbrace{f'(x_0)}_{\substack{\text{This is how} \\ \text{we denote the} \\ \text{limit, when it exists.}}}$$

exists.

The quantity $f'(x_0)$ is called the derivative of $f(x)$ at x_0. At the endpoints of the domain of $f(x)$, the derivative is defined using left/right limits and is called the left/right derivative of $f(x)$.

Start here.

Example: Show that $f(x) = |x|$ is not differentiable at $x_0 = 0$.

Solution: We need to show

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \quad \text{does not exist.}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

Using left and right limits:

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1, \quad \lim_{h \to 0^+} \frac{|h|}{h} = +1,$$

so the limit does not exist.

Definition: The derivative of $f(x)$ is a new function $f'(x)$ whose formula is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \left(\text{or left/right limit at end points}\right)$$

That is, the value of $f'(x)$ is the slope of $f(x)$ at any given point $x = x_0$.

Example: Show that if $f(x) = \frac{1}{x}$, then $f'(x) = \frac{-1}{x^2}$.

Solution: We compute

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$

<u>Remark</u>: The definition

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

can be written in another way. Substitute $x = x_0+h$, then $h = x - x_0$. Therefore as $h\to 0$, $x - x_0 \to 0$, meaning $x \to x_0$. Thus the limit above becomes

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0).$$

This is also occasionally used as the definition of the derivative.

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§2.2 Derivatives

Example: Show that if $f(x)=ax+b$, then $f'(x)=a$, using the definition of the derivative.

Solution: $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{a(x+h)+b-(ax+b)}{h}$$
$$=\lim_{h\to 0}\frac{ah}{h}=a.$$

In particular, as a special case (when $a=0$) we get:

The derivative of a ~~constart~~ constant is zero.

Example: Suppose that $n>0$ is a positive integer. Calculate $f'(x)$ if $f(x)=x^n$.

Proof: We find: $$\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{((x+h)-\cancel{x})[(\cancel{x}+h)^{n-1}+(x+h)^{n-2}x+\dots+x^{n-1}]}{h}$$

* Here we use the formula:

$$(a^n-b^n)=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+ab^{n-2}+b^{n-1})$$

with $a=x+h$ and $b=x$.

$$=\lim_{h\to 0}\underbrace{(x+h)^{n-1}+(x+h)^{n-2}x+\dots+x^{n-1}}_{n\text{ terms}}=nx^{n-1}.$$

So now we can differentiate some functions quickly, eg if $f(x) = x^3$ then $f'(x) = 3x^2$, or $f(x) = x^2 \Rightarrow f'(x) = 2x$.

Let's reflect on how much work these formula is letting us skip:

__Example__: Prove that $f'(x) = 2x$ when $f(x) = x^2$. Do not use any limit laws, use only ε-δ reasoning.

__Solution__: We must show

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = 2x$$

$$\Leftrightarrow \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = 2x.$$

I.e, use ε-δ to show $\lim_{h \to 0} 2x + h = 2x$, where h is the variable and x is regarded as a constant.

So let $\varepsilon > 0$. We must show that there exists $\delta > 0$ such that $0 < |h - 0| < \delta$ implies $|2x - h| < \varepsilon$.

So if $|h| < \delta$, then

$$|2x - h| \le |2x| + |h| < |2x| + \delta$$

So we must choose, for a fixed x, a δ so that

$$|2x| + \delta = \varepsilon \Rightarrow \delta = \varepsilon - |2x|.$$

Then check that such a δ works. Thus $f'(x) = 2x$.

<u>Notation</u>: If $y = f(x)$, there are many ways of writing what we have so far been writing as $f'(x)$.

$$f'(x) = y' = \frac{dy}{dx} = D_x y = D_x f = D f(x) = \frac{d}{dx} f(x).$$

The notation $\frac{dy}{dx}$ is very good in later math courses where you are dealing with multiple variables.

<u>Remarks</u>: ① $\frac{dy}{dx}$ is <u>not</u> a fraction, though there is a way of interpreting the symbols "dy" and "dx" that allows you to sometimes get away with manipulating $\frac{dy}{dx}$ like a fraction (See page 106, "Differentials").

② Evaluating a derivative at a given point is when you plug in a number: E.g. if $f'(x) = 2x$ (because $f(x) = x^2$) then $f'(3) = 6$ has a clear meaning. However if we're using the notation $\frac{dy}{dx} = 2x$, the way we would plug in $x = 3$ is by writing:

$$\left.\frac{dy}{dx}\right|_{x=3} = 6.$$

<u>Example</u>: Calculate $\left.\frac{dy}{dx}\right|_{x=2}$ if $y = \frac{x}{x^2+1}$.

Solution:

$$\frac{d}{dx}\left(\frac{x}{x^2+1}\right)\Bigg|_{x=2} = \lim_{h\to 0} \frac{\frac{2+h}{(2+h)^2+1} - \frac{2}{2^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2+h}{4+4h+h^2+1} - \frac{2}{5}}{h}$$

$$= \lim_{h\to 0} \frac{5(2+h) - 2(5+4h+h^2)}{5(5+4h+h^2)\cdot h}.$$

$$= \lim_{h\to 0} \frac{-3\cancel{h} - 2h^{\cancel{2}}}{5(5+4h+h^2)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-3-2h}{5(5+4h+h^2)}$$ (we can plug in h — it's a rational function with nonzero bottom)

$$= \frac{-3}{25}.$$

Next, we begin our study of derivative rules so that we can essentially put limits behind us — and yet use them all the time!

Derivative rules:

If $f(x)$ and $g(x)$ are differentiable at x, and C is a constant, then

(i) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

(ii) $(Cf(x))' = Cf'(x)$.

Example: Compute $f'(x)$ if $f(x) = 4x^3 - 2x + 1$.

Solution: Using the power rule and derivative sum/diff and constant rules:

$$
\begin{aligned}
(4x^3 - 2x + 1)' &= 4(x^3)' - 2(x)' + (1)' \\
&= 4(3x^2) - 2 + 0 \\
&= 12x^2 - 2.
\end{aligned}
$$