MATH 1230

§1.4 Continuity.

We saw already that some limits can be evaluated by simply "plugging in" numbers, like:

$$\lim_{x\to 2} x^2+5 = 2^2+5 = 9$$

Whereas others are trickier:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad (\text{can't plug in } 0, \text{ we'd get } \tfrac{0}{0}).$$

The functions whose limits are given by "plugging in" numbers are continuous functions.

Definition: A function f is continuous at a point c in the interior of its domain (meaning there is an open interval around c inside the domain) if

$$\lim_{x\to c} f(x) = f(c).$$

If $\lim_{x\to c} f(x)$ does not exist or exists but is not equal to $f(c)$, then $f(x)$ is discontinuous at c.

Example: If $f(x) = \dfrac{(x+1)(x-3)}{x-3}$, then

$\lim_{x\to 3} f(x) = \lim_{x\to 3} x+1 = 4$, but $f(3)$ does not exist

So $f(x)$ is not continuous at $x=3$.

Example: If $f(x) = H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

then $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$, so $\lim_{x \to 0} f(x)$ does not exist. Therefore $f(x)$ is not continuous at $x = 0$

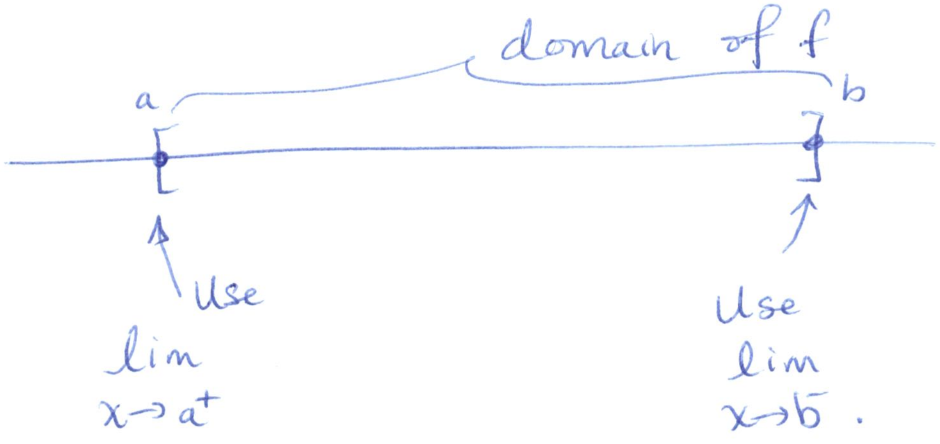
These ideas only apply to points in the interior of the domain of a function - for example, if the domain of $f(x)$ is $[a, b]$ then we're only talking about points in (a, b). The points a and b are dealt with by using left and right limits:

Definition: We say that $f(x)$ is right continuous at c if $\lim_{x \to c^+} f(x) = f(c)$, and left continuous if

$\lim_{x \to c^-} f(x) = f(c)$.

If c is the left end point of the domain of f, we say f is continuous at c if it is ~~left~~ right continuous there. If c is the right endpoint of the domain of f, we say f is continuous at c if it is left continuous there.

I.e.



If a function $f(x)$ is continuous at every point in its domain, we say it's continuous.

Many functions are continuous where they are defined:

(i) all polynomials

(ii) all rational functions

(iii) all functions $f(x) = x^{m/n}$

(iv) all trig functions

(v) absolute value: $f(x) = |x|$.

So any time you are asked to take a limit of such a function, you can simply plug in the value if the formula allows it.

Example: What is $\lim_{x\to -1} \dfrac{x^3-2x+1}{x^5-1}$?

Solution: It's a rational function, so

$$\lim_{x\to -1} \frac{x^3-2x+1}{x^5-1} = \frac{(-1)^3-2(-1)+1}{(-1)^5-1} = \frac{2}{-2} = -1.$$

Theorem: If f and g are continuous functions on some interval I, then the following are also continuous:

(i) $f \pm g$

(ii) fg

(iii) kf, $k \in \mathbb{R}$ any number

(iv) $\frac{f}{g}$ provided $g(x)$ is never 0.

(v) $(f)^{1/n}$, provided $f(x) > 0$ if n is even.

Last: If $g(x)$ is continuous at c and $f(x)$ is continuous at $g(c)$, then $f \circ g$ is continuous at c.

In particular this means:

$$\lim_{x \to c} f \circ g(x) = f\left(\lim_{x \to c} g(x)\right) = f(g(c)).$$

So you can "bring limits inside continuous functions".

Example: Evaluate $\displaystyle\lim_{x \to 2} \cos\left(\frac{x^2 + 2x + 1}{\sqrt{x}}\right)$.

Solution: Since cos is continuous

$$= \cos\left(\lim_{x \to 2} \frac{x^2+2x+1}{\sqrt{x}}\right)$$

$$= \cos\left(\frac{\lim_{x \to 2} x^2 + \lim_{x \to 2} 2x + \lim_{x \to 2} 1}{\lim_{x \to 2} \sqrt{x}}\right)$$

$$= \cos\left(\frac{2^2 + 2^2 + 1}{\sqrt{2}}\right) = \cos\left(\frac{9}{\sqrt{2}}\right).$$

So with all these rules, lots of practice and (in particular) a big bag of algebraic tricks, we can evaluate most limits that we come upon.

Example: Evaluate $\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$.

Solution: Since $\frac{1}{x-2} - \frac{4}{x^2-4}$ is not defined at $x=2$, we cannot plug in $x=2$. So we first do an algebraic trick:

$$\frac{1}{x-2} - \frac{4}{x^2-4} = \frac{x+2 \; - \; 4}{x^2-4} = \frac{x-2}{x^2-4}$$

So then

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2}$$

$$= \frac{1}{4}.$$

Now let us redo <u>all</u> of the limit concepts in a proper, mathematically rigorous manner.

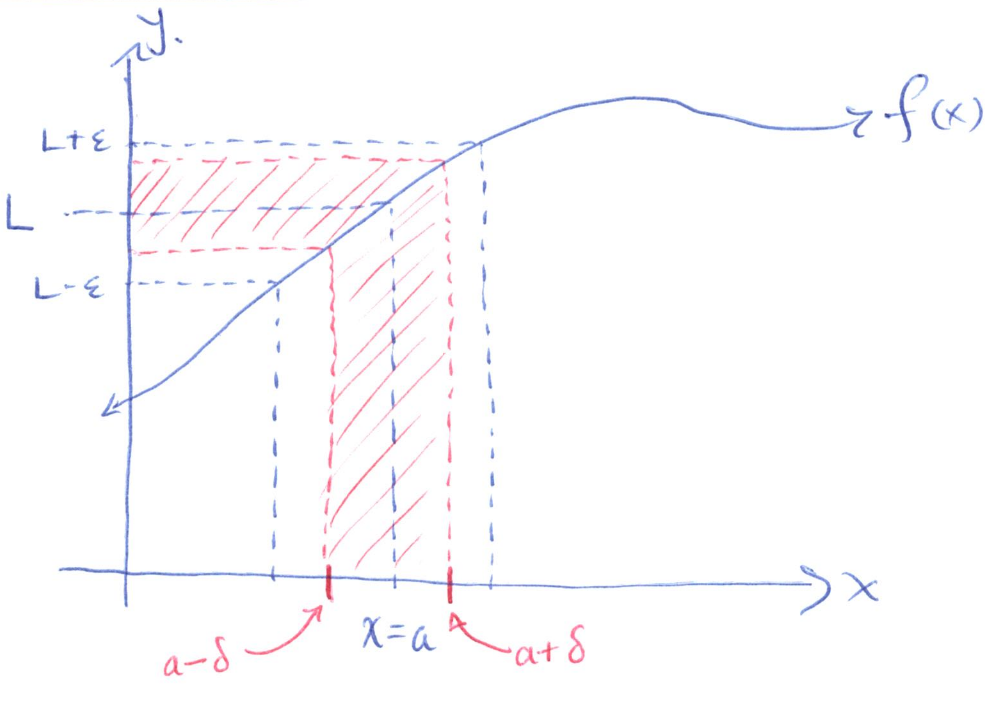
Here is the formal definition of a limit, which we will study next day:

<u>Definition</u>: The limit $\lim_{x \to a} f(x)$ exists and is equal to L if, for every positive distance $\varepsilon > 0$, we can guarantee that the values of $f(x)$ are within ε of L by choosing x to be within a distance $\delta > 0$ of a.

Another way of saying the same thing:

> For every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$ whenever x is in the domain of f.

<u>Picture</u>:



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§1.2 & 1.5 Limits revisited.

Recall the formal definition of a limit is:

<u>Definition</u>: The limit of $f(x)$ at $x=a$ is L if for every $\varepsilon>0$ there exists $\delta>0$ such that $0<|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$.

Let us take our first "real" limit.

<u>Example</u>: Prove that $\lim\limits_{x\to 2}(2x-1)=3$.

<u>Proof</u>: We want to show: For every $\varepsilon>0$, there exists $\delta>0$ such that $0<|x-2|<\delta$ implies $|(2x-1)-3|<\varepsilon$.

So suppose someone gives us an ε. We must show them how to find δ that satisfies

$$0<|x-2|<\delta \Rightarrow |(2x-1)-3|<\varepsilon$$

in order to convince them that such a δ always exists.

Well, observe that the inequality we want:

$$|(2x-1)-3|<\varepsilon$$

$$\Leftrightarrow |2x-4|<\varepsilon$$

$$\Leftrightarrow 2|x-2|<\varepsilon. \Leftrightarrow |x-2|<\frac{\varepsilon}{2}.$$

So if we choose $\delta = \frac{\varepsilon}{2}$, then it works. Our solution is:

Choose $\delta = \varepsilon/2$. Then whenever $0 < |x-2| < \frac{\varepsilon}{2}$, we have

$$|(2x-1)-3| = 2|x-2| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

So $\lim\limits_{x \to 2} (2x-1) = 3$.

Remark: In general, proofs of this type have two steps:

1. Showing how to pick δ, given ε
2. Showing your δ works.

Example: Prove that $\lim\limits_{x \to 3} x^2 = 9$.

Proof: We need: For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-3| < \delta \Rightarrow |x^2 - 9| < \varepsilon.$$

Finding δ:

First note that

$|x^2 - 9| = |x+3||x-3|$, and note that if $|x-3| < \delta$, then

$$|x+3||x-3| < \delta |x+3|.$$

So how big can $|x+3|$ be? We get

$$|x+3| = \underbrace{|x-3+6| \leq |x-3| + 6}_{\text{triangle inequality}} < \delta + 6 \quad \text{if } |x-3| < \delta.$$

Overall, if $|x-3|<\delta$ then

$|x^2-9| = |x+3||x-3| < (\delta+6)\delta = \delta^2+6\delta$. So we need to say how to choose δ so that $\delta^2+6\delta<\varepsilon$.

Trick: Suppose $\delta=1$ works, because $\varepsilon>7$. Then we're happy. Otherwise we know we'll have to choose $\delta<1$, in which case $\delta^2<\delta$ so that

$$\delta^2+6\delta < \delta+6\delta = 7\delta$$

and if we want $7\delta<\varepsilon$, we need $\delta<\frac{\varepsilon}{7}$.

So we choose $\delta = \min\{1, \varepsilon/7\}$.

Check δ works:

Let $\varepsilon>0$. Choose $\delta=\min\{1,\varepsilon/7\}$, and suppose $0<|x-3|<\delta$. Then $|x-3|<1$ and $|x-3|<\frac{\varepsilon}{7}$.

Since $|x-3|<1$, $x\in(2,4)$ so $x+3$ takes values in $(5,7)$, thus $|x+3|<7$. So

$|x^2-9| < |x+3||x-3| < 7\cdot\frac{\varepsilon}{7} = \varepsilon$. Thus $\lim_{x\to3} x^2 = 9$.

In many books, the labour behind choosing a correct δ is hidden, and they simply present an answer that works. Here is an example using right limits:

Example: Show that $\lim_{x\to0^+}\sqrt{x}=0$.

Proof: Let $\varepsilon>0$ be given. Choose $\delta=\varepsilon^2$.

Then if $|x-0| = |x| < \delta$ we get

$$|\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon.$$

So $\lim_{x \to 0^+} \sqrt{x} = 0$.

This type of proof is 100% correct and will get you full points. However it is best if you write down your reasoning behind your choice of δ, so that any reader of your work can understand your train of thought.

Example: Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Proof: We have to show that there is no value L such that $|f(x) - L| < \varepsilon$ whenever $|x - 0| = |x| < \delta$.

We'll suppose that there is such an L, and arrive at a contradiction.

So suppose $\lim_{x \to 0} \frac{|x|}{x} = L$, and someone challenges you to choose δ when $\varepsilon = \frac{1}{4}$. Suppose you find $\delta > 0$ so that $0 < |x - 0| < \delta$ implies $|f(x) - L| < \frac{1}{4}$.

Then look at plugging in $-\frac{\delta}{2}$ and $+\frac{\delta}{2}$. Both values $f\left(-\frac{\delta}{2}\right)$ and $f\left(+\frac{\delta}{2}\right)$ should be less than $\frac{1}{4}$ from L, whatever L is. But also $f\left(-\frac{\delta}{2}\right) = -1$ and $f\left(+\frac{\delta}{2}\right) = 1$, so overall:

$$2 = \left| f\left(\tfrac{\delta}{2}\right) - f\left(\tfrac{-\delta}{2}\right) \right| = \left| f\left(\tfrac{\delta}{2}\right) - L + L - f\left(\tfrac{\delta}{2}\right) \right|$$

$$\leq \left| f\left(\tfrac{\delta}{2}\right) - L \right| + \left| L - f\left(\tfrac{\delta}{2}\right) \right|$$

$$< \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

But $2 < \frac{1}{2}$ is a contradiction, so there must be no limit.

We can also revisit some of our old limit rules:

Claim: $\lim_{x \to a} c = c$ for any constant c.

Proof: Let $\varepsilon > 0$ be given. We need δ so that $0 < |x - a| < \delta$ implies $|f(x) - c| < \varepsilon$. But here, $f(x) = c$ for all x, so $|f(x) - c| = |c - c| = 0 < \varepsilon$. But $0 < \varepsilon$ is always true, regardless of the δ we choose. So any choice of δ will do.

Claim: $\lim_{x \to a} x = a$

Proof: Let $\varepsilon > 0$ be given. Choosing δ so that $0 < |x - a| < \delta$ implies $|f(x) - a| = |x - a| < \varepsilon$ means we need only choose $\delta = \varepsilon$. With this choice

$$|f(x) - a| = |x - a| < \delta = \varepsilon.$$

Claim: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

then $\lim_{x \to a} f(x) + g(x) = L + M$.

Proof: Let $\varepsilon > 0$ be given. Because $\lim_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies $|f(x) - L| < \varepsilon/2$.

Because $\lim_{x \to a} g(x) = M$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies $|g(x) - M| < \varepsilon/2$.

Set $\delta = \min\{\delta_1, \delta_2\}$.

Then $0 < |x-a| < \delta$ implies both $|f(x) - L| < \varepsilon/2$ and $|g(x) - M| < \varepsilon/2$. Then

$$
\begin{aligned}
|f(x) + g(x) - (L+M)| &= |f(x) - L + g(x) - M| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Our formal definition can also be "one-sided".

<u>Definition</u>: We say that $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $a < x < a + \delta$ and x belongs to the domain of f, then $|f(x) - L| < \varepsilon$.

Similarly for left-hand (ie. $\lim_{x \to a^-} f(x) = L$) limits.

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§1.3 + 1.5

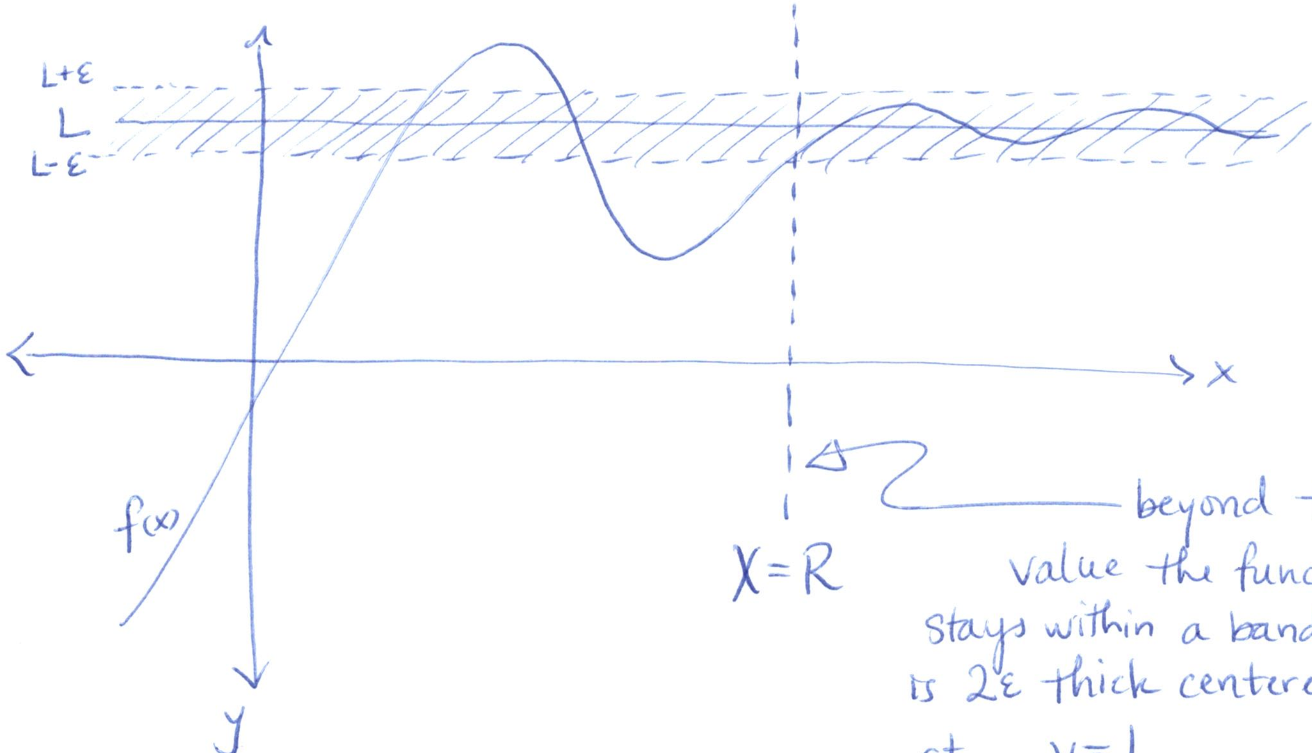
Last day we studied the formal definition of a limit:

$\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$.

We saw how limits are calculated for specific functions, and how the limit laws can be proved. Today, we re-do infinite limits.

Definition: We say that $\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ there exists a number R such that for every x in the domain of f

$$x > R \implies |f(x) - L| < \varepsilon.$$



beyond this value the function stays within a band that is 2ε thick centered at $y = L$.

__Example__: Prove that $\lim_{x\to\infty} \frac{2x+1}{x-3} = 2$

__Proof__: Let $\varepsilon>0$ be given. We need R so that $x>R$ implies $\left|\frac{2x+1}{x-3} - 2\right| < \varepsilon$.

So we need
$$\left|\frac{2x+1}{x-3} - 2\right| = \left|\frac{2x+1-2(x-3)}{x-3}\right|$$
$$= \left|\frac{7}{x-3}\right| = \frac{7}{|x-3|} < \varepsilon.$$

But $\frac{7}{|x-3|} < \varepsilon$ if and only if $|x-3| > \frac{7}{\varepsilon}$. If $R>3$ then $|x-3| = x-3$ and $|x-3|>\frac{7}{\varepsilon}$ becomes $x-3>\frac{7}{\varepsilon} \Rightarrow x > \frac{7}{\varepsilon}+3$. So if we choose $R = 3+\frac{7}{\varepsilon}$, it should work. (Check that it works!)

We also saw that some limits do not exist, like $\lim_{x\to 0} \frac{1}{x}$.

__Claim__: $\lim_{x\to 0} \frac{1}{x}$ does not exist.

__Proof__: Say $\lim_{x\to 0} \frac{1}{x} = L$ for some L. For $\varepsilon = 1$, there ought to be some $\delta>0$ such that $|x-0| = |x| < \delta$ implies $\left|\frac{1}{x} - L\right| < 1$.

In other words $L-1 < \frac{1}{x} < L+1$. But there is a problem here: choose $x>0$ to be smaller than δ and also smaller than $\frac{1}{L+1}$. Then $|x|<\delta$

but $x < \frac{1}{L+1} \Rightarrow L+1 < x$, contradicting $L-1 < \frac{1}{x} < L+1$.

This limit does not exist for a particular reason: The values of $f(x) = \frac{1}{x}$ become arbitrarily large as we approach 0. Formally:

Definition:

$\lim_{x \to a} f(x) = \infty$ means that for every $M>0$ there exists $\delta > 0$ s.t. $0<|x-a|<\delta \Rightarrow f(x) > M$.

Again, these limits can also be one sided:

Definition: $\lim_{x \to a^-} f(x) = -\infty$ means that for every $M<0$ there exists $\delta>0$ s.t. $a-\delta<x<a \Rightarrow f(x)<M$.

Remember: These limits still don't exist, we are just saying that there is a formal reason for their non-existence when we write $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

If $\lim_{x \to a} f(x)$, $\lim_{x \to a^-} f(x)$, or $\lim_{x \to a^+} f(x)$ is infinite, we say $f(x)$ has a vertical asymptote.

Example: If $f(x) = \frac{1}{x^2}$, prove that $\lim_{x \to 0} f(x) = \infty$.

Proof: Let $M > 0$ be any real number. Then $\frac{1}{x^2} > M$ provided $x^2 < \frac{1}{M}$, meaning we should choose $\delta = \frac{1}{\sqrt{M}}$. For if $\delta = \frac{1}{\sqrt{M}}$ then

$$0 < |x - 0| < \delta \Rightarrow 0 < |x| < \frac{1}{\sqrt{M}}$$

$$\Rightarrow |x|^2 < \frac{1}{M} \Rightarrow \frac{1}{|x|^2} > M.$$

However $|x|^2 = |x^2| = x^2$, so we've got $\frac{1}{x^2} > M$, as needed.

Example (How we actually evaluate these limits in practice).

Evaluate $\lim_{x \to 2^-} \frac{(x-5)}{(x-2)(x+3)}$.

Solution: As $x \to 2^-$ the denominator goes to zero while the top approaches $2 - 5 = -3$. So the quantity $\frac{x-5}{(x-2)(x+3)}$ becomes very large, though it could be either positive or negative. So we check signs:

If x is slightly to the left of 2, then

- $x - 5$ is slightly left of -3, so negative
- $x - 2$ is slightly left of 0, so negative

- $x+3$ is slightly left of 5, so positive.

Overall, two negatives and one positive = positive.

So $\lim_{x \to 2^-} \frac{x-5}{(x-2)(x+3)} = \infty$.

On the other hand, if $\lim_{x \to 2^+} \frac{x-5}{(x-2)(x+3)}$ is evaluated using the same reasoning, then for x slightly to the <u>right</u> of 2, $x-2$ is positive. So now $\lim_{x \to 2^+} \frac{x-5}{(x-2)(x+3)} = -\infty$.

We can also evaluate

$$\lim_{x \to -3^-} \frac{x-5}{(x-2)(x+3)} = -\infty \qquad \lim_{x \to -3^+} \frac{x-5}{(x-2)(x+3)} = +\infty$$

and from our old tricks:

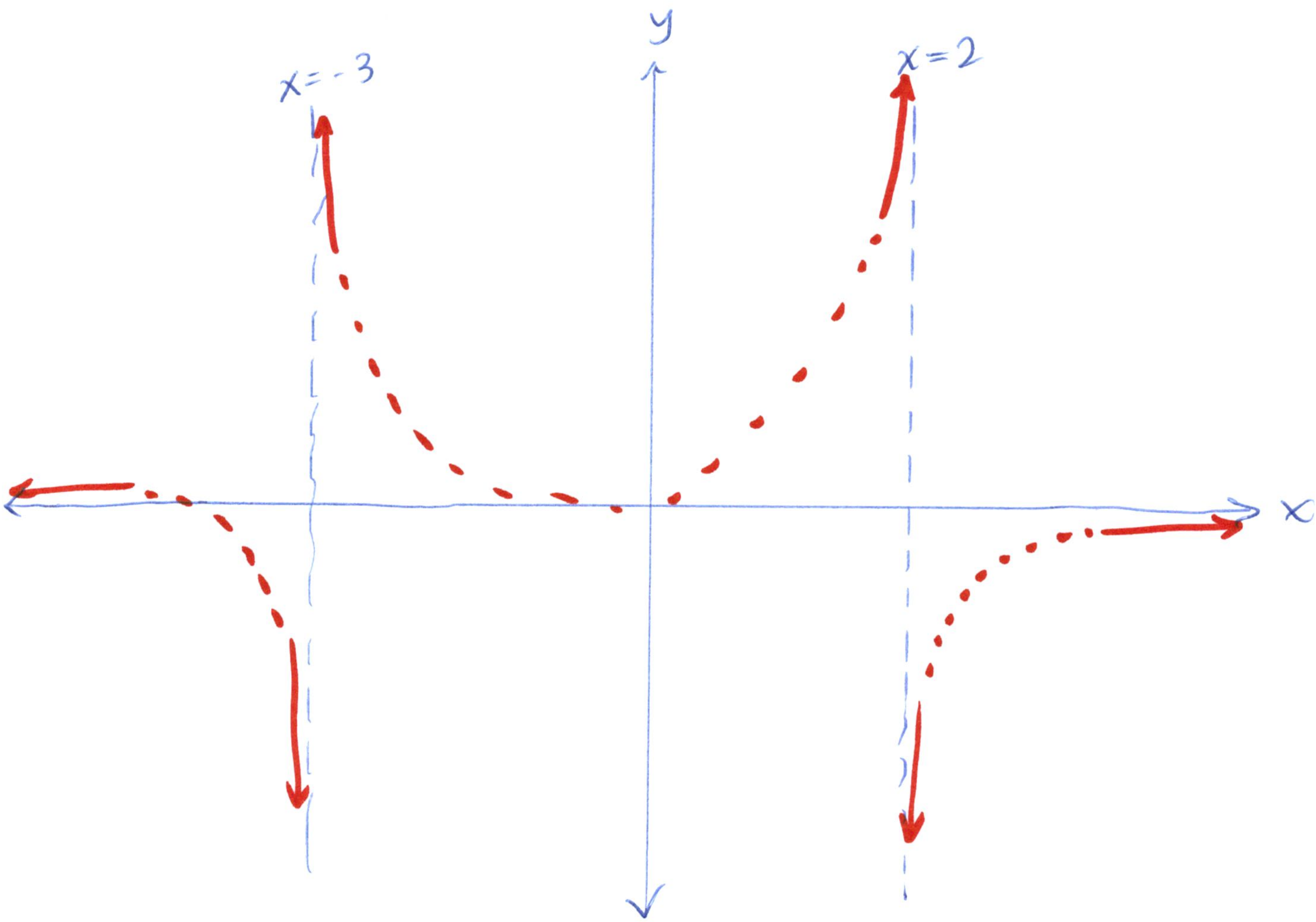
$$\lim_{x \to \infty} \frac{x-5}{(x-2)(x+3)} = \lim_{x \to \infty} \frac{x-5}{x^2+x-6} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} - 5/x^2}{1 + \frac{1}{x} - \frac{6}{x^2}}$$

$$= \frac{0-0}{1+0-0} = 0.$$

Similarly $\lim_{x \to -\infty} \frac{x-5}{(x-2)(x+3)} = 0$.

So we have vertical asymptotes at $x=2$ and $x=-3$, and horizontal one at $y=0$. Graphically we know this much about the function:



Here, the solid lines indicate our calculated limiting behaviour, the dotted lines are a guess. (We'll learn how to do better than guessing later on).