MATH 1230

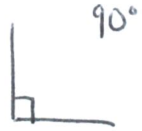
Section P7 Trig functions review.

We will measure all angles in <u>radians</u>.

One radian is the angle created when you walk a distance of 1 radius along the circumference of a circle:

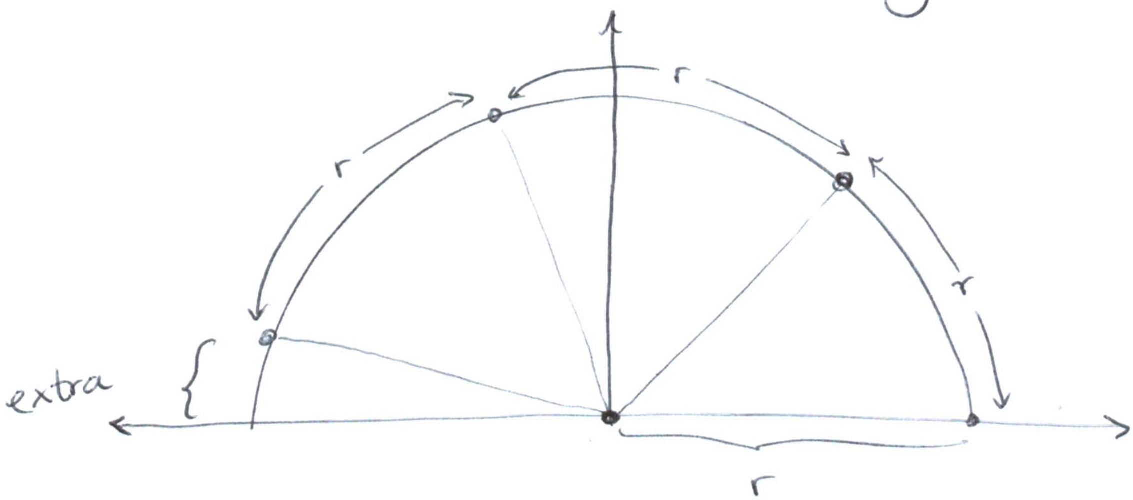
$\theta = 1$ radian.

Then common angles like



and

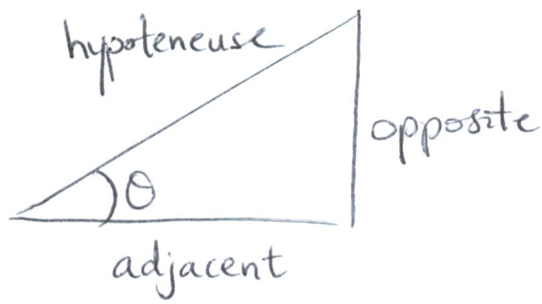
can all be realized in radians using the following fact:



$3r + \text{extra} = \pi$ radians. So $\llcorner = \pi/2$,

other common angles are $\frac{\pi}{4} = 45°$, $\pi/6 = 30°$, etc.

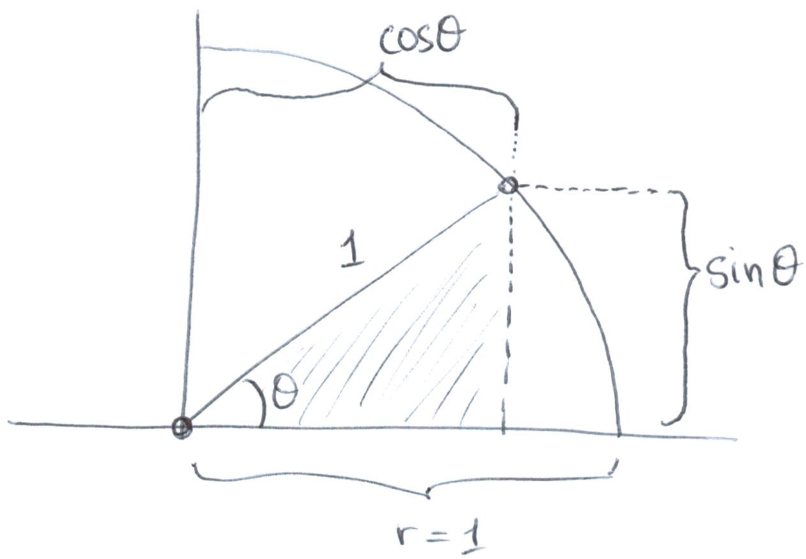
The functions sine, cosine and tangent all relate to triangles as follows:



Note:

$\tan = \dfrac{\sin}{\cos}$

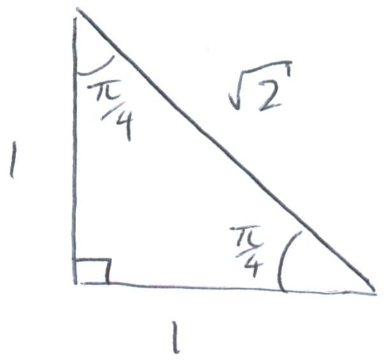
Then $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$, $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$, $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$. If the hypoteneuse has length 1, then you can think of this triangle as lying inside the unit circle in the x-y plane, and $(\cos(\theta), \sin(\theta))$ as the (x,y)-coordinates of the vertex of the triangle:

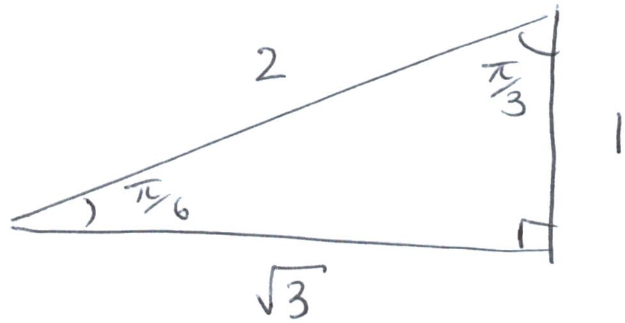


From this picture we can see why the identity $\cos^2\theta + \sin^2\theta = 1$ is true - it's the Pythagorean theorem.

Special values of trig functions:

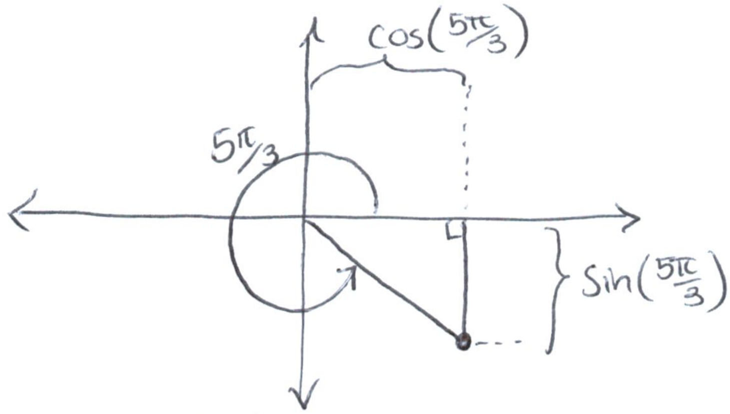
It is important to remember a few specific triangles and their side lengths in order to evaluate sin, cos and tan functions exactly for a few select values.





From these we can calculate sin, cos, tan of $\pi/6, \pi/4, \pi/3$.

What about quantities like $5\pi/3$? For this we draw $5\pi/3$ in the unit circle:



and we remember that $(\cos\theta, \sin\theta)$ are the (x, y) coordinates of the vertex of the triangle. We can see: $\cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and

$$\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

There are also many trig identities, some of which you should definitely know, such as:

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos\beta \pm \cos(\alpha)\sin(\beta)$$

These are useful, for example, in calculating exact values of trig functions.

Example: Evaluate $\cos\left(\frac{17\pi}{12}\right)$.

Solution: Note that $\frac{17\pi}{12} = \frac{20\pi}{12} - \frac{3\pi}{12} = \frac{5\pi}{3} - \frac{\pi}{4}$.

So
$$\begin{aligned}\cos\left(\frac{17\pi}{12}\right) &= \cos\left(\frac{5\pi}{3} - \frac{\pi}{4}\right) \\ &= \cos\left(\frac{5\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{3}\right)\sin\left(\frac{\pi}{4}\right). \\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \left(\frac{-\sqrt{3}}{2}\right)\cdot\frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\left(\frac{1-\sqrt{3}}{2}\right).\end{aligned}$$

Ideally, you should also know:

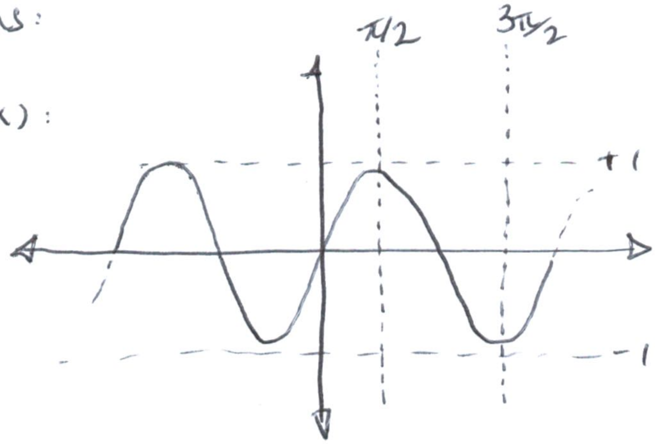
$$\cos^2 t = \frac{1+\cos 2t}{2} \qquad \sin^2 t = \frac{1-\cos 2t}{2}.$$

Less common than the functions sin, cos, tan are the functions:

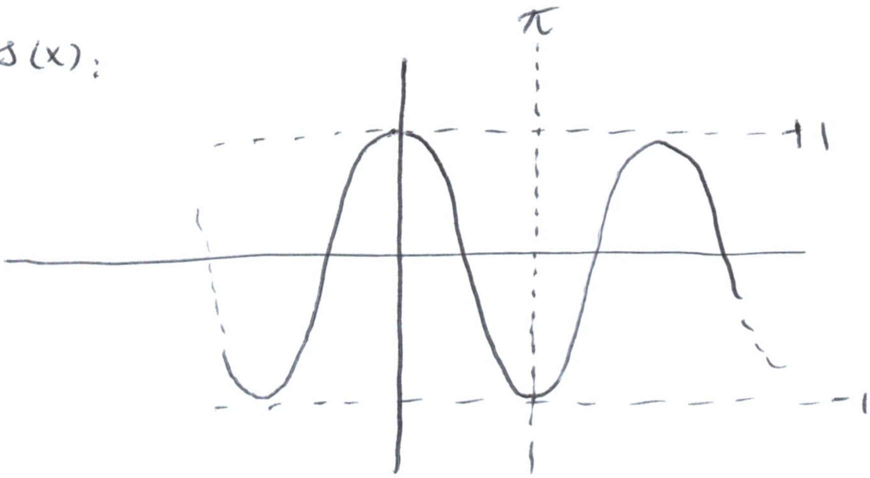
$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$

We will use these functions occasionally, so at least know their definitions. Should also know the following graphs:

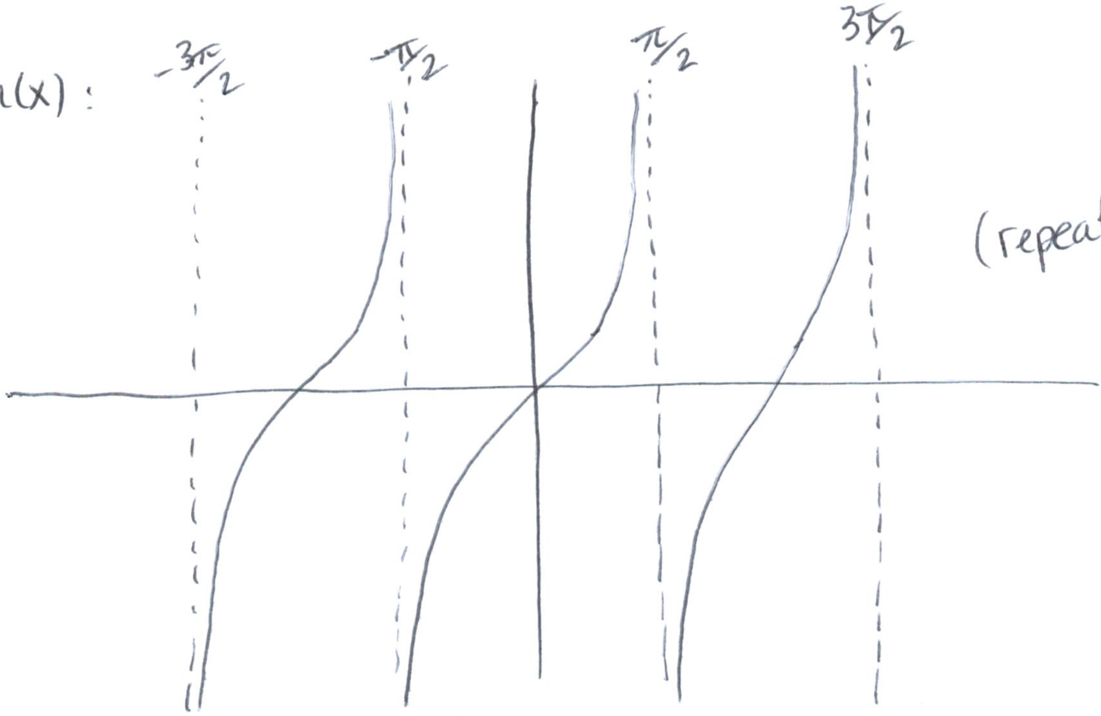
$\sin(x)$:



$\cos(x)$:



$\tan(x)$:



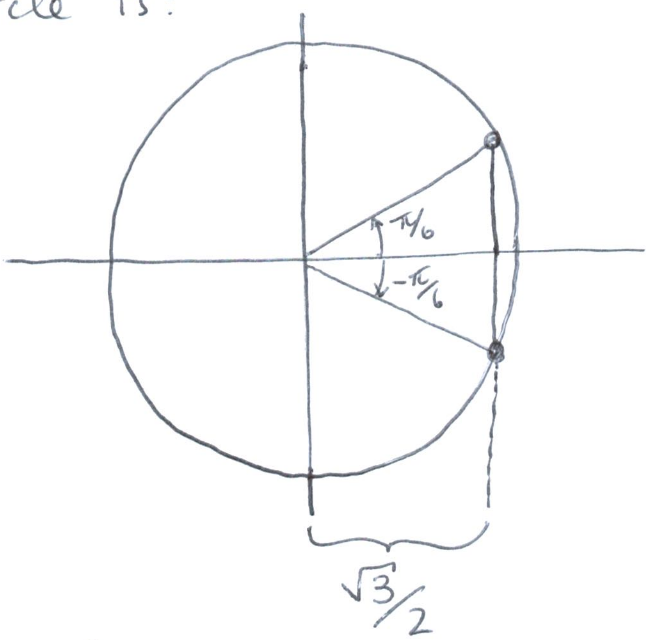
(repeating).

Example: Find all solutions of $\cos^2 2x = \frac{3}{4}$.

Solution: $\cos^2 2x = \frac{3}{4} \Rightarrow \cos 2x = \frac{\sqrt{3}}{2}$ or $\cos 2x = -\frac{\sqrt{3}}{2}$

① ②

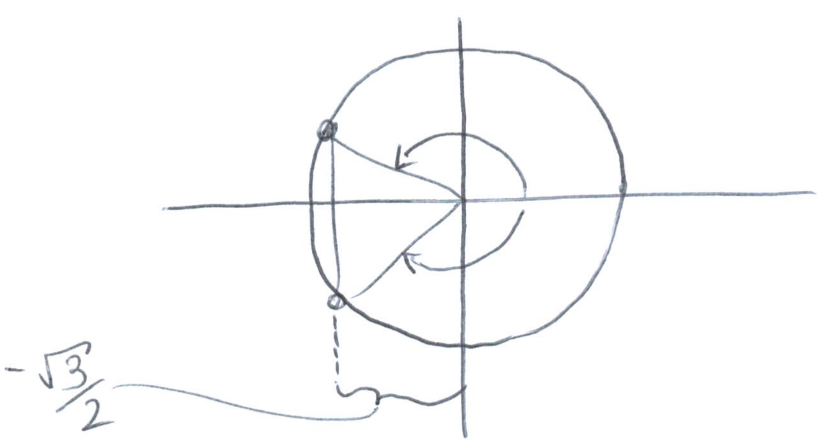
In case ①, $\cos 2x = \frac{\sqrt{3}}{2}$ means that our picture on the unit circle is:



so obvious solutions are $2x = \pm\frac{\pi}{6}$, of course cosine repeats every 2π so we get $2x = \pm\frac{\pi}{6} + 2k\pi$, k any integer.

$\Rightarrow x = \pm\frac{\pi}{12} + k\pi$, k any integer.

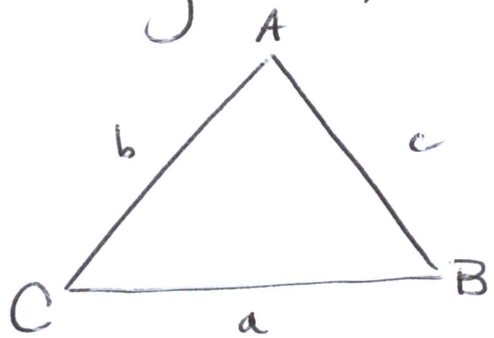
In case ②, our picture on the unit circle is:



So we see that $2x = \pm \frac{5\pi}{6} + 2k\pi$, k any integer

$$\Rightarrow x = \pm \frac{5\pi}{12} + k\pi, \ k \text{ any integer.}$$

There are also trig identities that will be used less frequently that you should be aware of (at least know they exist). Given a triangle:



Then we have:

Sine Law: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Law: $a^2 = b^2 + c^2 - 2bc\cos A$

You can think of the cosine law as the Pythagorean theorem for triangles <u>without</u> a right angle, and "$-2bc\cos A$" is the "correction term".

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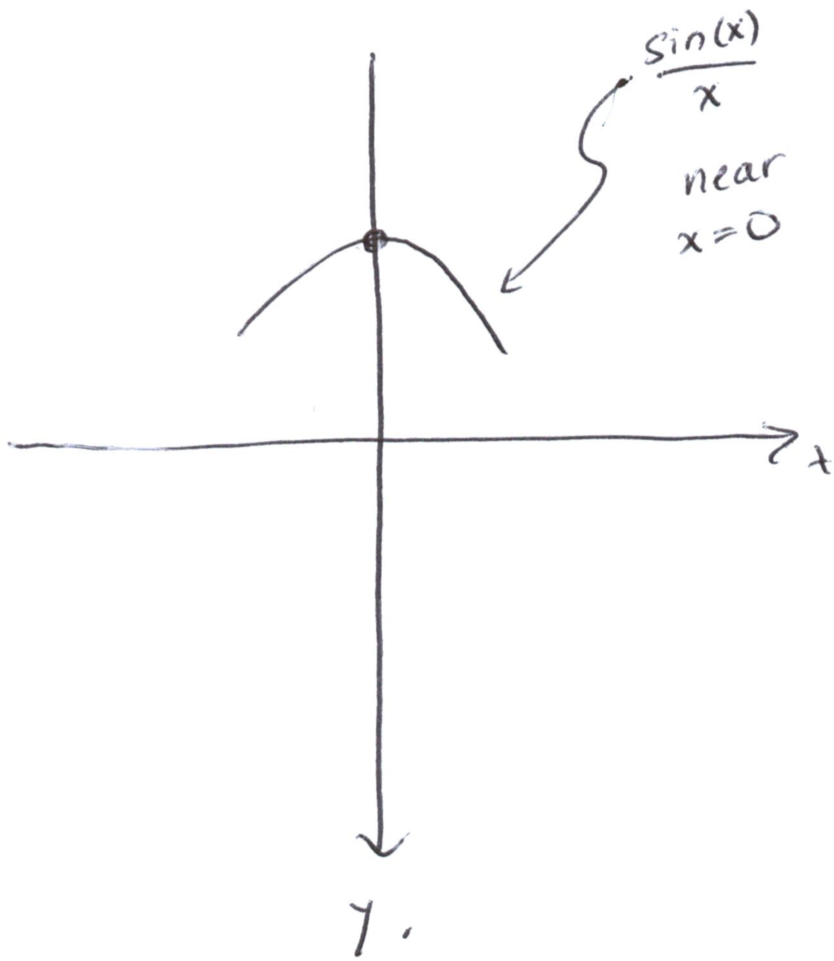
§1.2 Introduction to limits.

This section will be our first informal look at limits. We will re-visit them and give a formal definition next week.

Suppose we are graphing the function $\frac{\sin x}{x} = f(x)$ by creating a table of values. At $x=0$, we cannot plug in $x=0$ to get a value for $f(x)$, as the formula is undefined there. So we plug in values very near to 0:

x (radians)	$f(x)$
-0.7	0.92031
-0.2	0.99347
-0.05	0.99958
0	undefined
0.01	0.99998
0.03	0.99985
0.3	0.98506
1.4	0.70389



So we see that as the values of x that we plug in to $f(x)$ approach zero (from either side), the outputs of $f(x)$ approach 1. This is a limit!

We write $\lim_{x \to 0} f(x) = 1$.

Informal definition of a limit:

If $f(x)$ is defined ~~near~~ for all x near $x=a$, except possibly at a itself, and if we can make the output of $f(x)$ as close to some number L as we please by plugging in values close to a, then

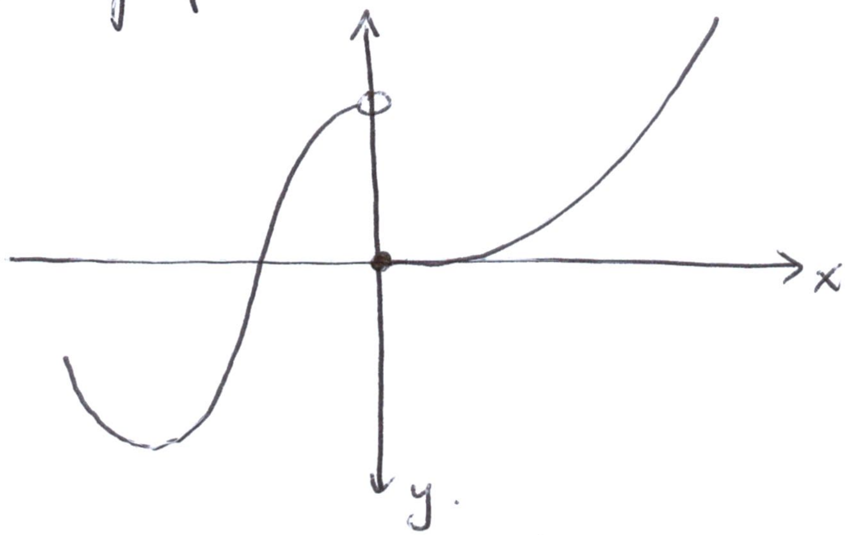
$$\lim_{x \to a} f(x) = L.$$

Remark: This definition is heuristic and not good enough to do serious mathematics! However it is good enough for us to investigate a few basic properties of limits, like how to evaluate them in simple cases.

Example: Consider

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ \cos(x) & \text{if } x < 0. \end{cases}$$

Then the graph of $f(x)$ is as below near $x=0$:



Note that plugging in x-values near 0 gives two possibilities for $f(x)$:

- $f(x)$ is close to 1 if we're to the left of 0
- $f(x)$ is close to 0 if we're to the right of 0.

So there is no single number L that the outputs are close to. Therefore we say

$\lim_{x \to 0} f(x)$ does not exist.

Example: Suppose $f(x) = \dfrac{x^2+2x+1}{x+1}$. Then for $x \neq -1$, $f(x) = \dfrac{(x+1)^2}{x+1} = x+1$, whereas for $x=-1$, the formula is undefined. Since limits involve the values near but not equal to a certain x-value, we can say:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} x+1$$

But now $x+1$ is just a line of slope 1, so we know its value at $x=-1$. Therefore

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} x+1 = 0.$$

Aside from canceling common factors, there's one more significant trick at this point.

Example: Evaluate $\displaystyle\lim_{x \to 4} \frac{\sqrt{x}-2}{x^2-16}$.

Solution: At $x=4$ the expression is undefined.

Trick: When you see $(\sqrt{a} \pm b)$, multiply by the conjugate $\sqrt{a} \mp b$. (both top & bottom)

Here we get $\displaystyle\lim_{x \to 4} \frac{\sqrt{x}-2}{x^2-16} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$

$$= \lim_{x \to 4} \frac{x-4}{x^2-16\,(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(x+4)(\sqrt{x}+2)} \qquad \text{(common factor trick)}$$

$$= \lim_{x \to 4} \frac{1}{(x+4)(\sqrt{x}+2)} = \frac{1}{32}$$

at this point we can plug in $x=4$ since the formula is now defined there.

One-sided limits:

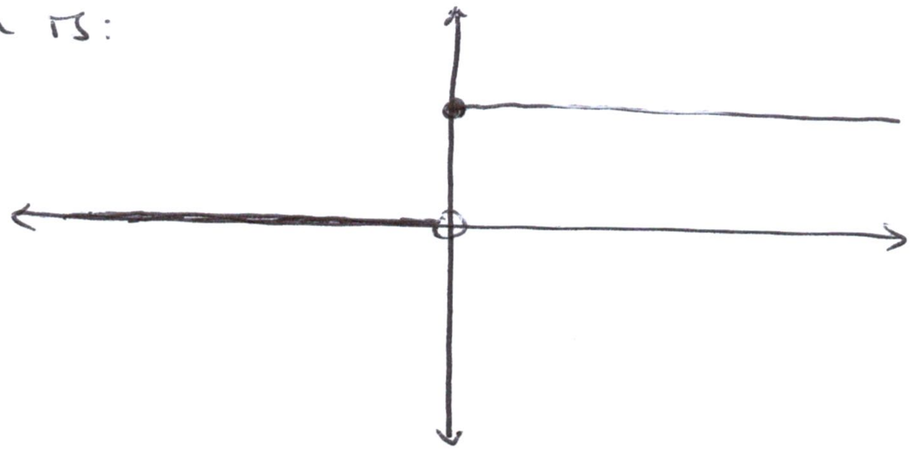
When the values of $f(x)$ approach different numbers depending on whether we plug in x-values to the left or right of some number $x=a$, the limit does not exist.

To study the behaviour of a function $f(x)$ at $x=a$ from one side only we restrict our "informal definition" to consider x values from only one side of a, and we arrive at left and right limits.

Example: Recall

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

The graph is:



From the left, as $t \to 0$ $H(t) \to 0$. We write

$$\lim_{t \to 0^-} H(t) = 0$$

minus sign means only consider t to the left

From the right, as $t \to 0$ $H(t) \to 1$, so

$$\lim_{t \to 0^+} H(t) = 1$$

means values to the right of 0.

Example: $\lim_{x \to 0^-} \lfloor x \rfloor = \begin{cases} \lfloor x \rfloor & \text{if } x \text{ is not an integer} \\ \lfloor x \rfloor - 1 & \text{if } x \text{ is an integer} \end{cases}$

One-sided limits are related to "regular" limits in the following way:

Theorem:

A function $f(x)$ has a limit at $x=a$ if and only if it has both left and right limits at $x=a$ and the left and right limits are equal. In this case:

$$\lim_{x \to a^-} = \lim_{x \to a^+} = L \quad \text{if and only if} \quad \lim_{x \to a} f(x) = L.$$

Example: If $f(x) = \dfrac{|x-2|}{x^2+x-6}$, calculate

$\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, and $\lim_{x \to 2} f(x)$.

Solution: We do two cases. If $x > 2$, then

$|x-2| = x-2$ so

$$\lim_{x\to 2^+} \frac{|x-2|}{x^2+x-6} = \lim_{x\to 2^+} \frac{x-2}{(x-2)(x+3)}$$

$$= \lim_{x\to 2^+} \frac{1}{x+3} = \frac{1}{5}.$$

If $x<2$ then $|x-2| = -x+2$, so

$$\lim_{x\to 2^-} \frac{|x-2|}{x^2+x-6} = \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+3)}$$

$$= \lim_{x\to 2^-} \frac{-1}{x+3} = -\frac{1}{5}.$$

Since these limits are not equal, $\lim\limits_{x\to 2} f(x)$ does not exist.

Example (Theorem)

If $p(x)$ is any polynomial then

$$\lim_{x\to a} p(x) = p(a).$$

If $q(x)$ is another polynomial and $q(a) \neq 0$, then

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

<u>Note that this will not happen in general</u> !

If we can evaluate the limit of a function simply by plugging in the value $x=a$, then the function is very ~~spec~~ special, in a way we will soon make precise.

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§1.2–1.3: Limit rules, squeeze theorem and infinite limits.

Last day we saw an "informal definition" of limits and a few tricks for evaluating limits, also left vs right limits.

Here is a list of limit rules for evaluating limits of sums, products, etc of functions:

Limit rules: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then:

① $\lim_{x\to a} c = c$ for any constant c

② $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$.

③ $\lim_{x\to a} f(x)g(x) = LM$

④ $\lim_{x\to a} c\cdot f(x) = cL$ for any constant c

⑤ $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$

⑥ $\lim_{x\to a} \left(f(x)\right)^{m/n} = L^{m/n}$ whenever $L^{m/n}$ exists.

For dealing with trickier functions, we also have:

Theorem (Squeeze Theorem)

Suppose $f(x) \le g(x) \le h(x)$ holds in an interval (c, d) containing a. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

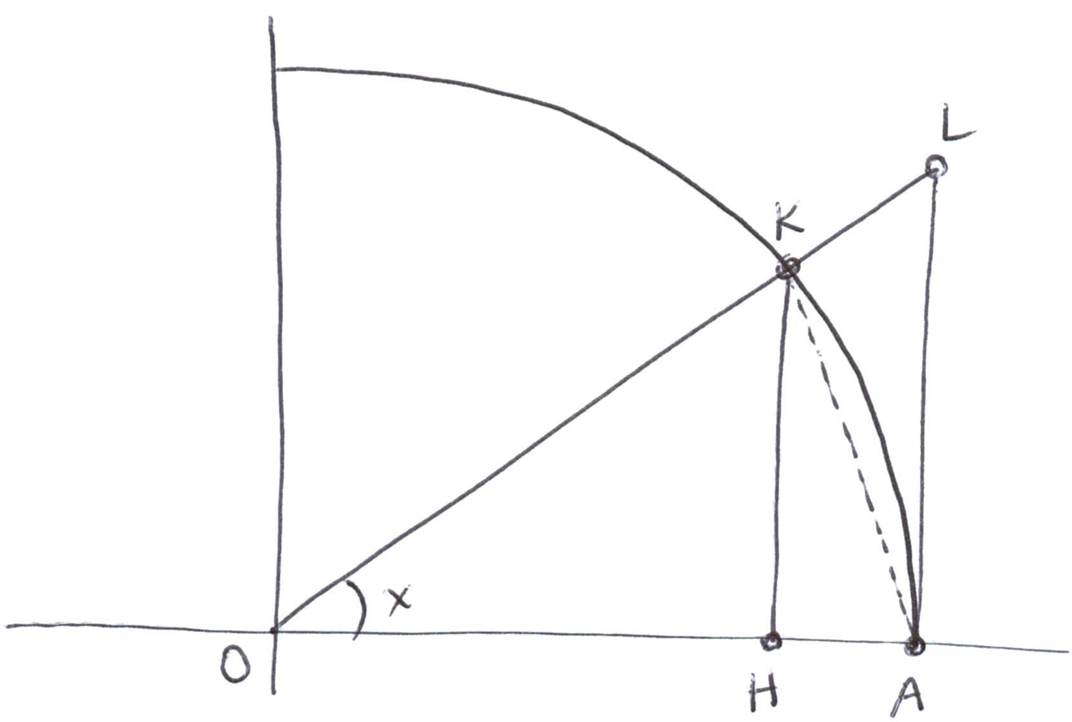
Then $\lim_{x \to a} g(x) = L$ also.

Example: Use the squeeze theorem to prove

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Solution: We will show $\cos(x) \le \frac{\sin x}{x} \le 1$ near $x=0$.

By symmetry (all functions involved are even) it suffices to look at $(0, \frac{\pi}{2})$. Let $x \in (0, \pi/2)$ and consider:



Then

$$\text{Area}(\Delta KOA) \le \text{Area of sector } KOA \le \text{Area}(\Delta LOA)$$

$$\frac{\sin(x)}{2} \le \frac{x}{2} \le \frac{\tan(x)}{2}$$

Multiply by $\frac{2}{\sin(x)} > 0$:

$$1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$$

Take reciprocals:

$$1 \ge \frac{\sin(x)}{x} \ge \cos(x).$$

Then taking limits:

$$1 = \lim_{x\to 0} 1 \ge \lim_{x\to 0} \frac{\sin(x)}{x} \ge \boxed{\lim_{x\to 0} \cos(x) = 1}$$

We will see why this is true shortly

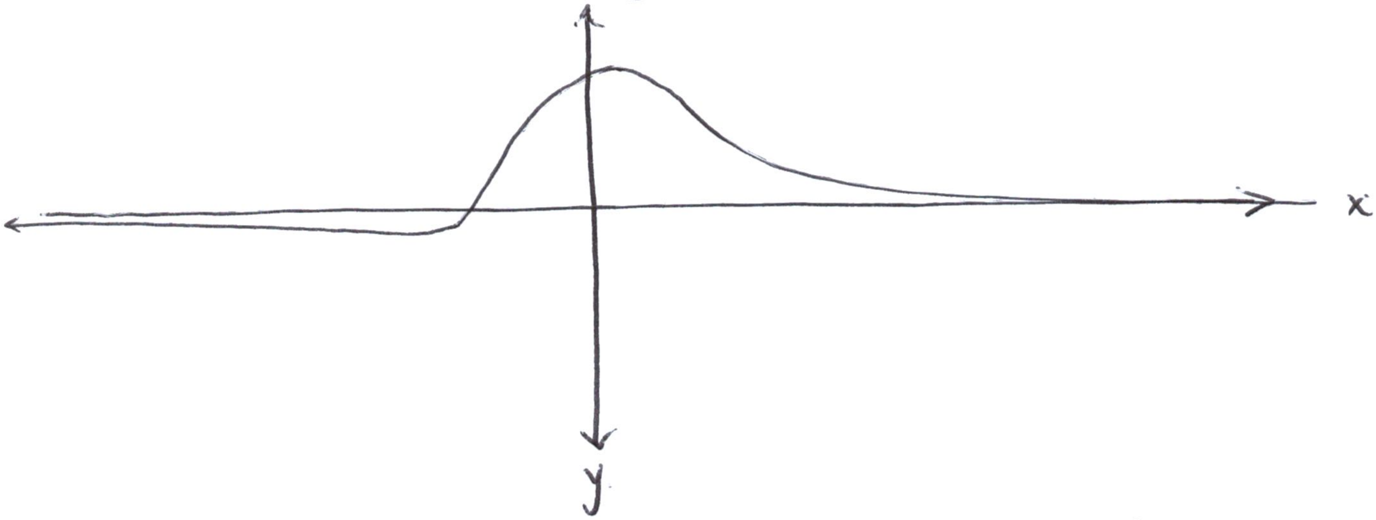
$$\Rightarrow \lim_{x\to 0} \frac{\sin(x)}{x} = 1,$$ by the squeeze theorem.

§1.3. Infinite limits.

Consider the function

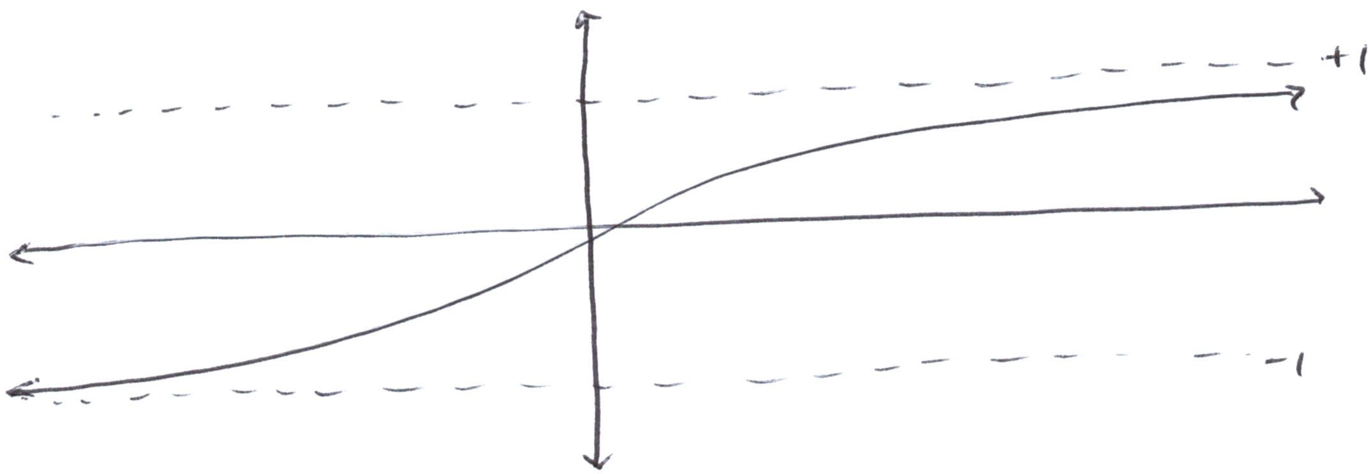
$$f(x) = \frac{x+2}{x^2+1}.$$

The graph is roughly:



So, for x very large, $f(x)$ is near 0; for x very large and negative, $f(x)$ is near 0 as well.

Similarly, if $f(x) = \dfrac{x}{\sqrt{x^2+1}}$



and $f(x) \to 1$ as $x \to \infty$, $f(x) \to -1$ as $x \to -\infty$.

We write:

$$\lim_{x \to -\infty} f(x) = -1 , \quad \lim_{x \to +\infty} f(x) = 1 .$$

Infinite limits (informal definition)

If $f(x)$ is defined on (a,∞) for some a, and if we can make $f(x)$ as close to L as we please by choosing x very large and positive, then

$$\lim_{x\to\infty} f(x) = L.$$

On the other hand if $f(x)$ is defined on $(-\infty, a)$ and $f(x)$ is as close to M as we please when we choose x very large and negative, then

$$\lim_{x\to -\infty} f(x) = M.$$

Example: A trick for rational functions:

Evaluate $\displaystyle\lim_{x\to\infty} \frac{2x^2-3x+5}{x^2+1}$.

Solution:

$$\lim_{x\to\infty} \frac{2x^2-3x+5}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2-\frac{3}{x}+\frac{5}{x^2}}{1+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{3}{x} + \lim_{x\to\infty}\frac{5}{x^2}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{2+0+0}{1+0} = 2.$$ Note we used $\lim\limits_{x \to \pm\infty} \frac{1}{x^n} = 0$ for $n > 0$.

<u>Example</u>: Compute

$$\lim_{x \to -\infty} \frac{\sqrt{5x^2-1}}{x+6}$$

<u>Solution</u>: Here we also divide through by powers:

$$\lim_{x \to -\infty} \frac{\sqrt{5x^2-1}}{x+6} \cdot \frac{\frac{1}{x}}{1/x}$$

$$= \lim_{x \to \infty} \frac{-\sqrt{\frac{1}{x^2}}\sqrt{5x^2-1}}{\frac{1}{x}(x+6)}$$

(Here we need to bring $\frac{1}{x}$ under the square root, it becomes $\frac{1}{x^2}$ and we get a "$-$" in front since $x<0$)

$$= \lim_{x \to -\infty} \frac{-\sqrt{5x^2/x^2 - 1/x^2}}{1 + 6/x}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{5 - 1/x^2}}{1 + 6/x} = \frac{-\sqrt{5}}{1} = -\sqrt{5}$$

Another kind of "infinite limit" is like

$$\lim_{x \to 0} \frac{1}{x^2} = \infty \quad \text{or} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

This does not mean the limit exists at $x=0$.

It means that the values of $f(x)$ become arbitrarily large and positive as

- x approaches 0 from either side, in the case of $\lim_{x\to 0} \frac{1}{x^2} = \infty$, or

- x approaches 0 from the left, in the case of $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

We can also have limits which go to ∞ as $x \to \infty$, for example

$\lim_{x\to\infty} x^2+1 = \infty$, meaning x^2+1 grows without bound as $x\to\infty$.

Example: Calculate $\lim_{x\to -\infty} \frac{x^3+1}{x^2+1}$.

Solution: Again, divide by powers (highest power in the denominator!)

$$\lim_{x\to -\infty} \frac{x^3+1}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to -\infty} \frac{x + 1/x^2}{1 + 1/x^2}$$

$$= \left(\lim_{x\to -\infty} \left(x + 1/x^2\right)\right) \Big/ 1 = -\infty$$