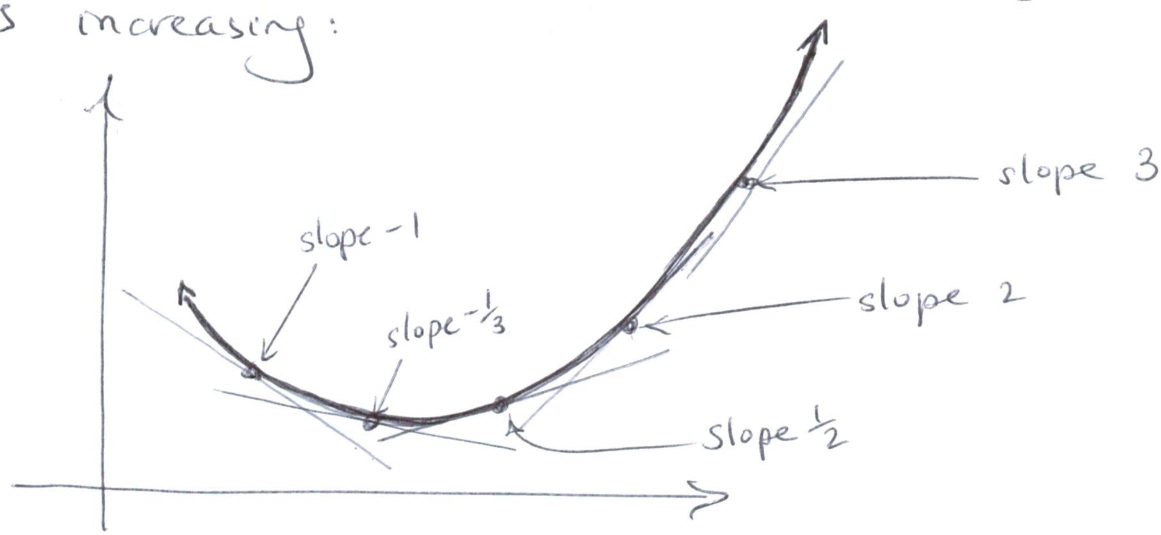
MATH 1230

§4.5. Concavity and inflections (ie. the graphical meaning of the second derivative)

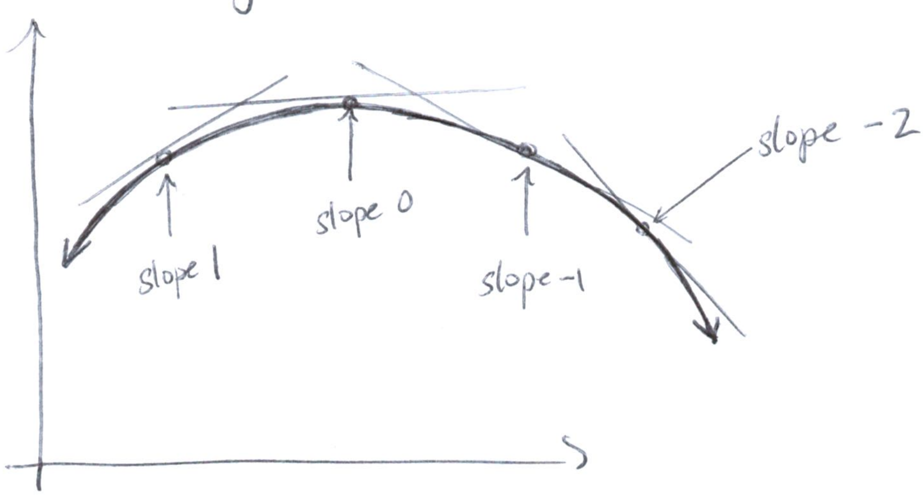
The first derivative is the slope of the tangent line. The second derivative is therefore <u>the rate of change of the slope of the tangent line</u>.

So, if $f''(x) > 0$ then the slope of the tangent line is increasing:



we call this type of curve <u>concave up</u>.

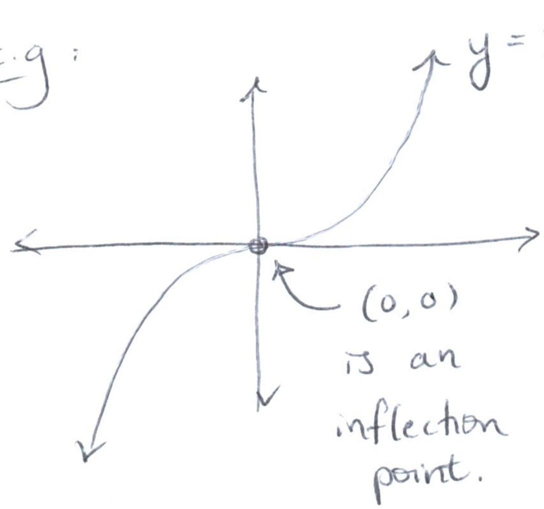
If $f''(x) < 0$ then the slope of the tangent line is decreasing:

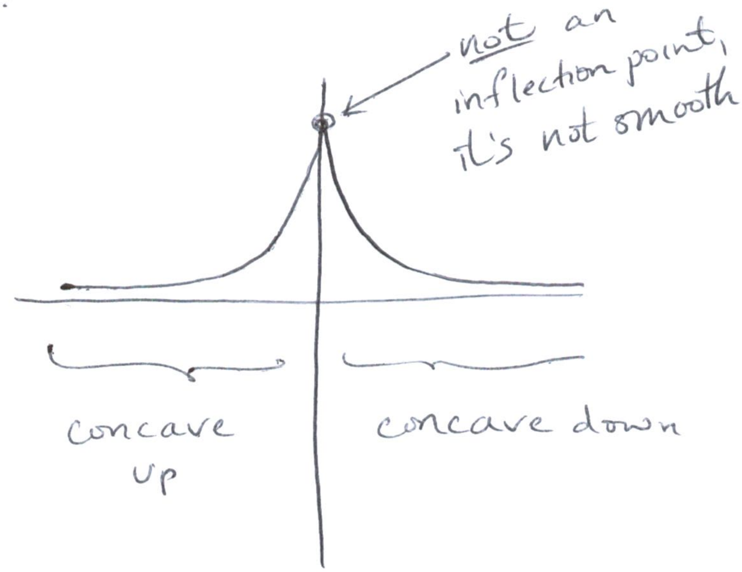


we call this type of curve <u>concave down</u>.

The place where a "smooth" graph changes from being concave down to concave up, or vice versa, is called an inflection point:

E.g:





<u>Specifically</u>: We say that $(x_0, f(x_0))$ is an inflection point of the curve $y=f(x)$ if

(i) The graph $y=f(x)$ has a tangent line at $x=x_0$ (it's "smooth"), and

(ii) The concavity of $f(x)$ changes at $x=x_0$.

<u>Remark</u>: This means that when $(x_0, f(x_0))$ is an inflection point and $f''(x_0)$ exists, it must be zero.

<u>Example</u>: Determine the intervals of concavity and inflection points of $f(x) = 2x - \sin^{-1}(x)$.

<u>Solution</u>: We need the second derivative to do this analysis. We compute:

$$f'(x) = 2 - \frac{1}{\sqrt{1-x^2}} = 2 - (1-x^2)^{-\frac{1}{2}}$$

Therefore $f''(x) = \frac{1}{2}(1-x^2)^{-3/2} \cdot 2x$, on $[-1,1]$ since the domain of our original function is $[-1,1]$. To determine where $f''(x)$ may change sign, we look for:

(i) points where $f''(x) = 0$, and

(ii) points where $f''(x)$ is undefined.

First, $f''(x) = 0 \Rightarrow (1-x^2)^{-3/2} = 0$ $\underline{\text{or}}$ $2x = 0$

$\Rightarrow x = \pm 1$ $\qquad\qquad \Rightarrow x = 0.$

There will be no sign change at ± 1 because these are the endpoints of the domain. On the other hand, $f''(x)$ changes sign at $x=0$ because:

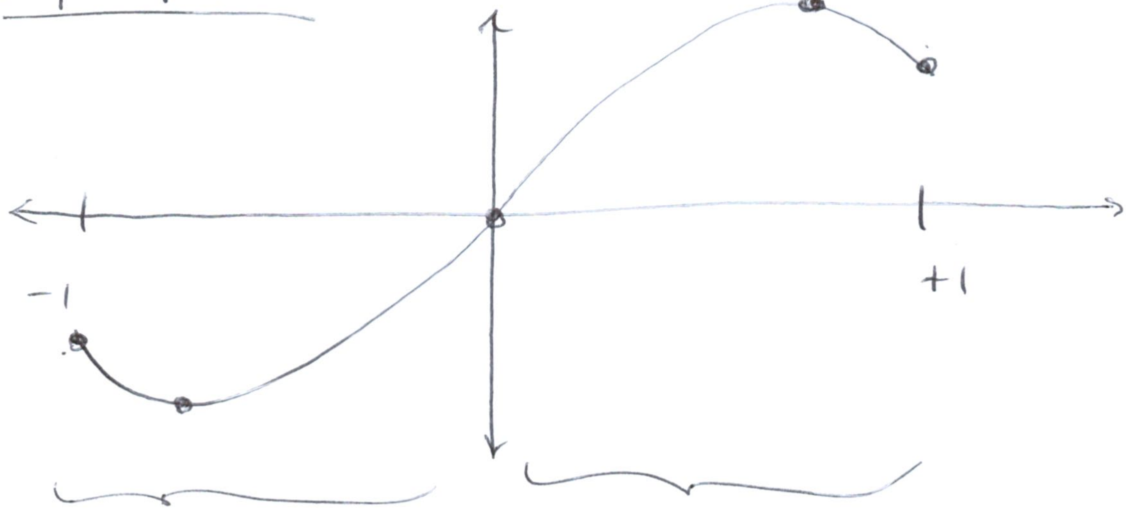
If $x < 0$, $f''(x) = \underbrace{\frac{1}{\sqrt{(1-x^2)^3}}}_{\text{pos}} \cdot \underset{\text{neg}}{x} < 0$

if $x > 0$ $f''(x) = \underbrace{\frac{1}{\sqrt{(1-x^2)^3}}}_{\text{pos}} \cdot \underset{\text{pos}}{x} > 0.$

So $f(x)$ is concave down on $(-1, 0)$ and concave up on $(0, 1)$. Since $f'(x) = 2 - \frac{1}{\sqrt{1-x^2}}$ exists at $x=0$, the point $(0, f(0)) = (0, 0)$ is an inflection point of $f(x)$.

Note from last day we had calculated:

Graph of $f(x)$:



use a concave up curve here.

use a concave down curve here.

Next class we'll explore curve sketching in depth.

Last day, we discussed how to find critical points and decide whether or not the are local maxes or mins. We called our method "the first derivative test". There's also a "second derivative test":

Theorem: If $f'(x_0)=0$, then

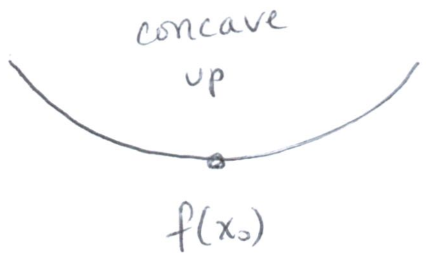
a) If $f''(x_0)>0$ then $f(x)$ has a local minimum at x_0

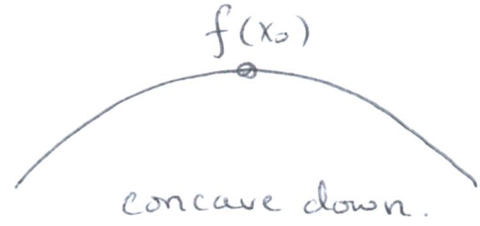
b) If $f''(x_0)<0$ then $f(x)$ has a local maximum at x_0.

c) If $f''(x_0)=0$ we cannot draw any conclusion.

Roughly, this is true because:

$f''(x_0) > 0$ $\qquad\qquad\qquad\qquad$ $f''(x_0) < 0$





But there's a proof!

<u>Proof</u>: Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$ (so we're in case b). Then consider

$$\lim_{h \to 0} \frac{f'(x_0+h)}{h} = \lim_{h \to 0} \frac{f'(x_0+h) - \overset{0}{\overset{\shortparallel}{f'(x_0)}}}{h} = f''(x_0) < 0.$$

Since this limit exists and is negative, we know

$\lim_{h \to 0^+} \frac{f'(x_0+h)}{h} < 0$, meaning for small positive h we must have $f'(x_0+h) < 0$

and $\lim_{h \to 0^-} \frac{f'(x_0+h)}{h} < 0$, meaning for small negative h we must have $f'(x_0+h) > 0$.

So f' changes sign from positive to negative as we travel left to right past x_0. Therefore $f(x)$ has a local max, by the first derivative test.

<u>Example</u>: Find and classify the critical points of

$$f(x) = x^2 e^{-x}.$$

Solution: If $f(x) = x^2 e^{-x}$, then

$$f'(x) = x^2(-e^{-x}) + 2xe^{-x}$$
$$= (2x - x^2)e^{-x}$$

And therefore $f'(x) = 0 \longrightarrow \underline{2x - x^2 = 0}$ or $\boxed{e^{-x} = 0}$ impossible

$$\Downarrow$$
$$x(2-x) = 0$$

so $x = 0$ or $x = 2$.

Now we use the second derivative test:

$$f''(x) = (2x - x^2)' e^{-x} + (2x - x^2)(-e^{-x})$$
$$= (2 - 2x)e^{-x} + (x^2 - 2x)e^{-x}$$
$$= e^{-x}(x^2 - 4x + 2)$$

Thus $f''(0) = e^0(0 - 4 \cdot 0 + 2) = 2 > 0$, so $x = 0$ gives a local min by the second derivative test, and

$f''(2) = e^2(4 - 8 + 2) = e^2 \cdot (-2) < 0$, so $x = 2$ gives a local max by the second derivative test.

Remark: The second derivative test is often easier than the first derivative test, but it can fail for some fairly straightforward functions.

For example, if $f(x) = x^4$ then $f'(x) = 4x^3$ and so $f'(x) = 0 \Rightarrow x = 0$ is a critical value, and we know it gives a minimum. However if we were to try and apply the second derivative test, we would get

$$f''(x) = 12x^2, \text{ so } f''(0) = 12 \cdot 0 = 0.$$

Therefore the test is inconclusive.

Or, a little different: If $f(x) = \sin(x)$ then $f'(x) = \cos(x)$, so $f'(x) = 0$ at $x = \frac{\pi}{2} + k\pi$. However $f''(x) = -\sin(x)$, and $f''(\frac{\pi}{2} + k\pi) = \sin(\frac{\pi}{2} + k\pi) < 0$ for all k. So the second derivative test fails again for every critical value of $f(x) = \sin(x)$.

§ 4.5 Curve sketching. Questions 1–40

Sketching a curve is a multi step process. We saw some of the most important steps last day:

- Finding where $f(x)$ is increasing, decreasing
- Finding where $f(x)$ is concave up/down
- Using first or second derivatives to identify local maxes and mins.

In General:

A) what is the domain of $f(x)$?

B) If it's easy to solve, solve $f(x)=0$ to get x-intercepts. what is the y-intercept $f(0)$?

C) Is the function even? Odd? Does it repeat like $\sin(x)$?

D) Are there horizontal or vertical asymptotes? (Take $\lim_{x \to \pm\infty} f(x)$ here).

E) Where is it inc/decr

F) Find max/min.

G) Where is it concave up/down?

H) MAKE THE SKETCH!

Example: Sketch $y = x^{5/3} - 5x^{2/3}$.

Solution:

A) Here, note that $x^{5/3} = \sqrt[3]{x^5}$ and $x^{2/3} = \sqrt[3]{x^2}$.

There are no problems with taking cube roots of negatives, so the domain is all of $\mathbb{R}$.

B) Intercepts.

The y-intercept is

$$y(0) = 0^{5/3} - 5 \cdot 0^{2/3} = 0$$

So the curve passes through $(0,0)$.

The x-intercept is

$$x^{5/3} - 5x^{2/3} = 0$$

$$\Rightarrow x^{2/3}(x-5) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 5.$$

So the x-intercept is $x=0$, $x=5$.

c) Symmetry.

Plugging in $-x$ for x gives

$$y(-x) = (-x)^{5/3} - 5(-x)^{2/3}$$

$$= \sqrt[3]{(-x)^5} - 5\sqrt[3]{(-x)^2}$$

$$= \sqrt[3]{-x^5} - 5\sqrt[3]{x^2} = -x^{5/3} - 5x^{2/3}$$

since this isn't equal to either $y(x)$ or $-y(x)$, the function is not even or odd.

D) Asymptotes. Vertical

The function has no vertical asymptotes, the function never goes to $\pm\infty$.

Horizontal: $\lim_{x\to\infty}(x^{5/3}-5x^{2/3}) = \lim_{x\to\infty} x^{5/3}(1-5x^{-1})$

$$= \lim_{x\to\infty} x^{5/3} \cdot \lim_{x\to\infty}\left(1-\frac{5}{x}\right) = \infty$$

$$\lim_{x\to-\infty}(x^{5/3}-5x^{2/3}) = \lim_{x\to\infty} x^{5/3}\left(1-\frac{5}{x}\right) = -\infty$$

So there are no horizontal asymptotes.

E) Increasing / Decreasing

We calculate
$$y' = \frac{5x^{2/3}-10x^{-1/3}}{3}$$

so the critical values $y'=0$ are $\frac{5x^{-1/3}(x-2)}{3} = 0$.

I.e. $\frac{5}{\sqrt[3]{x}} \cdot \frac{x-2}{3} = 0$.

So $x=2$ is a critical value since $y'=0$,

$x=0$ is also a critical value since y' undefined.

Make a quick table:

function	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$x^{-1/3}$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
$y' = f'(x)$	$+$	$-$	$+$
y	$\nearrow$ incr.	$\searrow$ decr.	$\nearrow$ incr.

F) Maxes and mins:

$x=0$ gives a max where $y = 0^{5/3} - 5\,0^{2/3} = 0$

$x=2$ gives a min where $y = 2^{5/3} - 5 \cdot 2^{2/3}$
$= -3 \cdot 2^{2/3}$

G) Concavity / inflection pts.

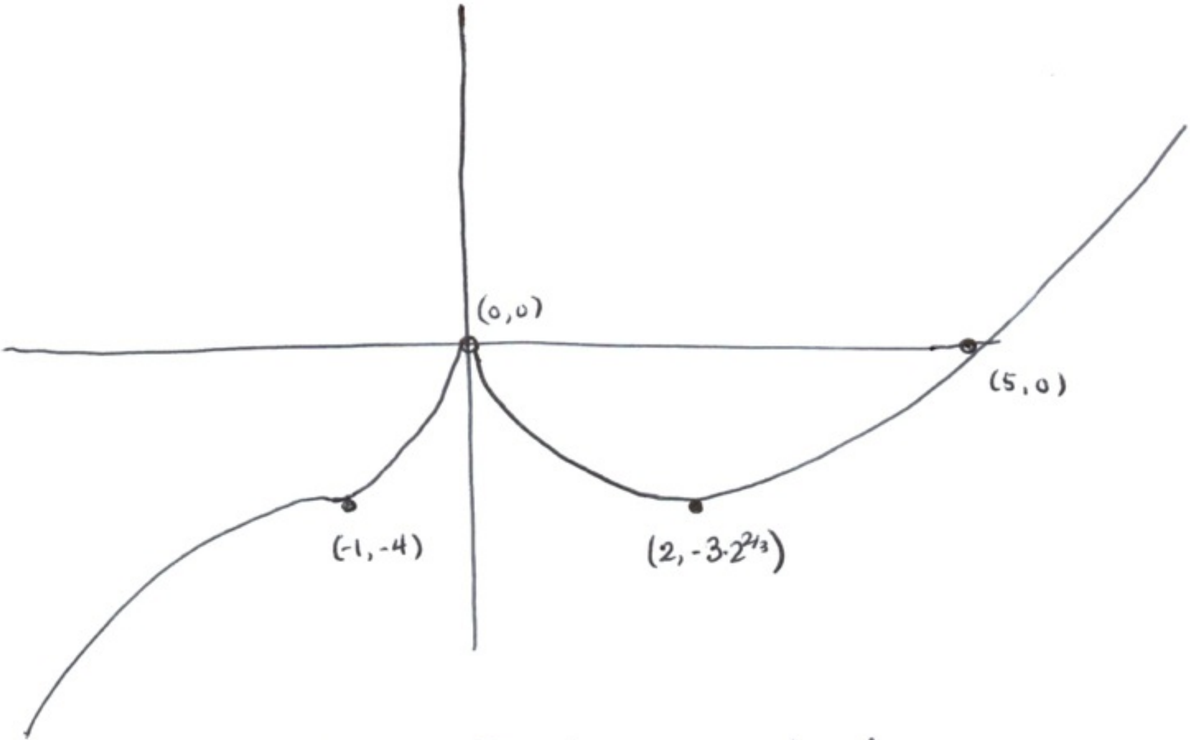
$y''(x) = \dfrac{10x^{-4/3}(x+1)}{9}$, so y'' is undefined at $x=0$ and has a root at $x=-1$.

So $x=0$ and $x=-1$ are potential inflection points.

$\frac{1}{\sqrt[3]{x^4}}$ = function	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
$x^{-4/3}$	$+$	$+$	$+$
$x+1$	$-$	$+$	$+$
$y''(x)$	$-$	$+$	$+$
$y(x)$	$\frown$ conc. down	$\smile$ conc. up	$\smile$ conc. up.

So when $x=-1$ $y=-4$ is an inflection point.

H) Sketch.



Why a point at $x=0$? You can check

$$\lim_{x\to 0^-} y'(x) = +\infty \quad \text{and} \quad \lim_{x\to 0^+} y'(x) = -\infty.$$

Example. Sketch $y = x\sqrt{2-x^2}$.

Solution:

A) We need $2-x^2 \geq 0$ or $x^2 \leq 2$, so x is between $-\sqrt{2}$ and $\sqrt{2}$.

B) y-intercept is $y(0) = 0 \cdot \sqrt{2-0^2} = 0$.

x-intercepts are $x \cdot \sqrt{2-x^2} = 0$

$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}.$$

C) We plug in $(-x)$

$$y(-x) = -x\sqrt{2-(-x)^2} = -x\sqrt{2-x^2} = -y(x),$$

so y is odd.

D) There are no vertical asymptotes because y does not go to infinity anywhere.

There are no horizontal asymptotes because x is between $-\sqrt{2}$ and $\sqrt{2}$, so $x \to \pm\infty$ is not possible.

E) Increasing / Decreasing

We calculate
$$y' = \frac{2-2x^2}{\sqrt{2-x^2}} = 2\,\frac{(1-x)(1+x)}{\sqrt{2-x^2}}$$

It is undefined at $x = \pm\sqrt{2}$, the endpoints of the domain. It is zero for
$$2-2x^2=0$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1.$$

function	$(-\sqrt{2},-1)$	$(-1, 1)$	$(1, \sqrt{2})$
$(1-x)$	+	+	−
$(1+x)$	−	+	+
$y'(x)$	−	+	−
$y(x)$	↘	↗	↘
	decr.	incr.	decr.

F) There is a min at $x=-1$ where
$$y = (-1)\sqrt{2-1^2} = (-1)\cdot 1 = -1$$
and a max at $x=1$ where $y = 1\cdot\sqrt{2-1^2} = 1.$

G) The second derivative is

$$y'' = \frac{3x^3 - 6x}{(2-x^2)^{3/2}}$$

This gives inflection points by solving $y''=0$

$$\Rightarrow 3x^3 - 6x = 0 \Rightarrow 3x(x^2-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \pm\sqrt{2}$$

So $x=0$ is the only potential inflection point since $\pm\sqrt{2}$ are the endpoints of the domain.

Note that

$(2-x^2)^{3/2}$ is always positive, it's the square root of something, and

(x^2-2) is positive when x is in $(-\sqrt{2}, \sqrt{2})$.

So $y'' = \dfrac{3x(x^2-2)}{(2-x^2)^{3/2}}$ is neg when $x<0$

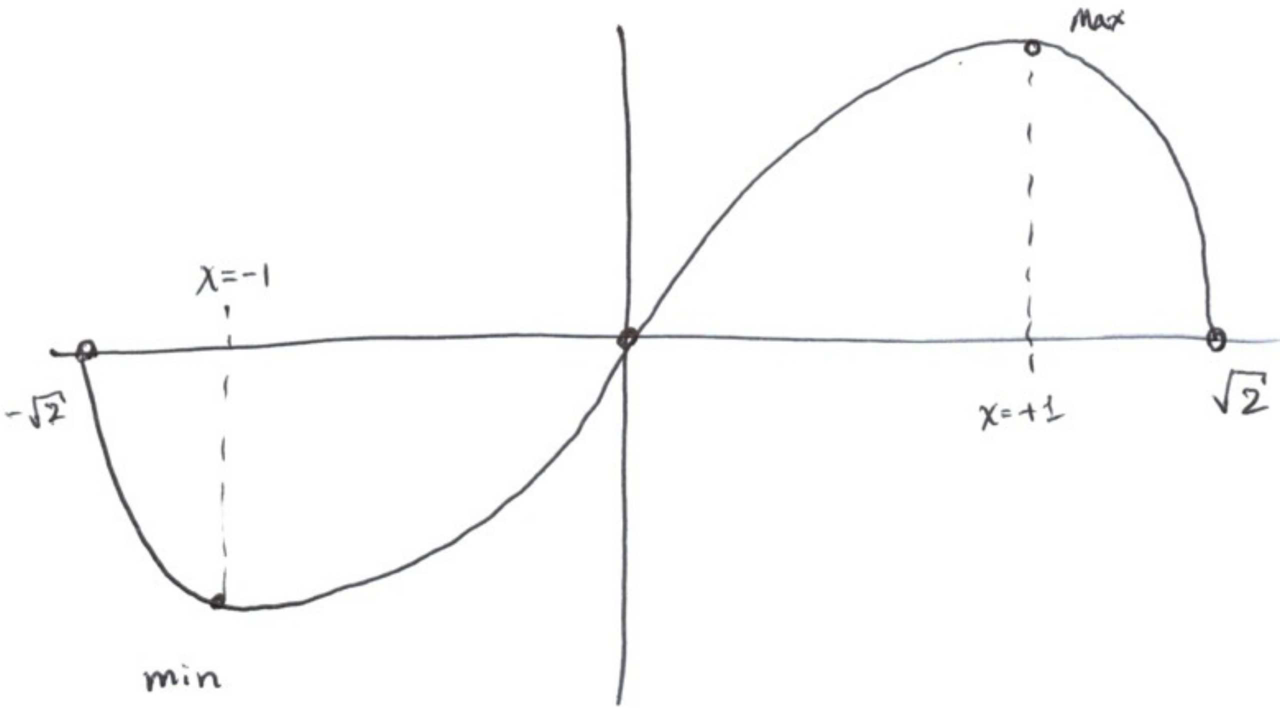
y'' is pos when $x>0$

$\Rightarrow$ $y(x)$ is concave down on $(-\sqrt{2}, 0)$

$y(x)$ is concave up on $(0, \sqrt{2})$.

$(0,0)$ is an inflection point.

H) Sketch.


Max
x=-1
-√2
x=+1
√2
min

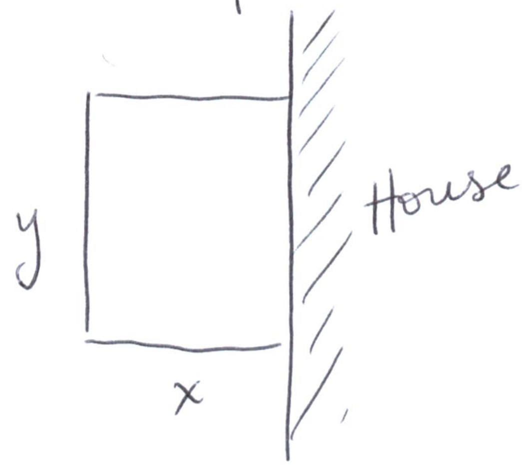
MATH 1230

§4.8 Extreme Value Problems.

In this section we apply our procedure for finding max/min values of functions to real-world scenarios.

Example: You buy 15m of fencing to build a rectangular enclosure alongside your house. Find the maximum possible area of the enclosure.

Solution: The picture is



So Area $= xy$

and

Perimeter $= 2x + y$,

but we know the perimeter must be 15m.

To maximize the area we need a function of one variable, so use

$$15 = 2x + y \Rightarrow y = 15 - 2x$$

to eliminate y. Then

$$A = (15 - 2x)x = 15x - 2x^2.$$

Now we note: The domain of this function A is $[0, \frac{15}{2}]$, because we cannot have the side lengths of the enclosure be less thank 0!

Now since A is continuous on $[0, \frac{15}{2}]$ it must attain a max there, so we check critical points and endpoints.

So solve $A' = 15 - 4x = 0 \Rightarrow x = \frac{15}{4}$; and also consider the endpoints $x = 0$ and $x = \frac{15}{2}$ of the domain. We get:

$$A(0) = 0$$

$$A\left(\tfrac{15}{4}\right) = 15\left(\tfrac{15}{4}\right) - 2\left(\tfrac{15}{4}\right)^2 = \frac{15^2}{8}$$

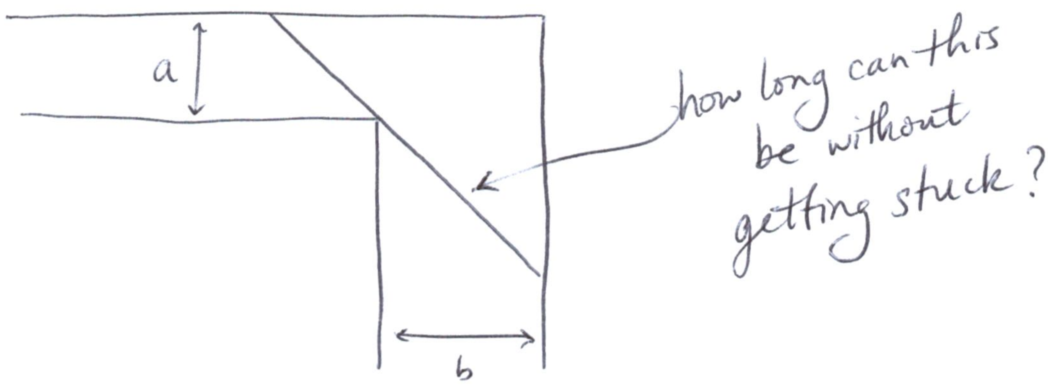
$$A\left(\tfrac{15}{2}\right) = 0.$$

So when $x = \frac{15}{4}$ we get the max area. So the dimensions we want are $x = \frac{15}{4}$, $y = \frac{15}{2}$.

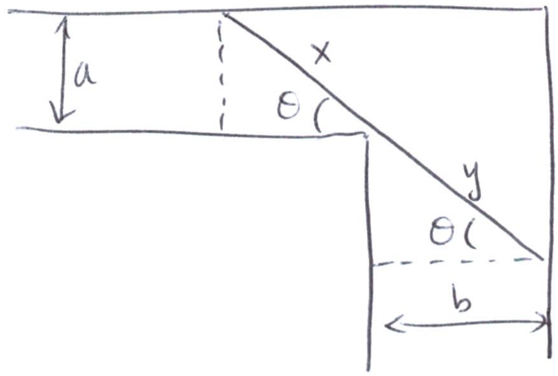
These problems can get very hard!

<u>Example</u>: Find the length of the longest beam that can fit horizontally around a corner from a hallway of width 'a' to a hallway of width "b".

<u>Solution</u>: The picture is:



Label the elements of the problem:



Write down the equations that arise as a result of our labels: $\sin\theta = \frac{a}{x}$, $\cos\theta = \frac{b}{y}$, and $\text{length} = x + y$.

Thus, for every angle θ in $[0, \frac{\pi}{2}]$ as we go around the corner, we must have

$$\text{length} = x + y = \frac{a}{\sin\theta} + \frac{b}{\cos\theta} = a\csc\theta + b\sec\theta.$$

For some value of θ, this length will be at a minimum — meaning that we're at the "tightest spot" in our trip around the corner, and no larger beam will fit. So we must minimize

$$L = \text{length} = a\csc\theta + b\sec\theta \quad \text{where } \theta \text{ is in } [0, \tfrac{\pi}{2}].$$

$$\Rightarrow \quad L' = -a\csc\theta\cot\theta + b\sec\theta\tan\theta$$

Now we solve for critical values where $L' = 0$, and check endpoints in order to minimize it. First, solve

$$0 = -a\csc\theta\cot\theta + b\sec\theta\tan\theta$$

$$\Rightarrow \frac{a}{b} = \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta}$$

$$= \frac{\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}}{\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}}$$

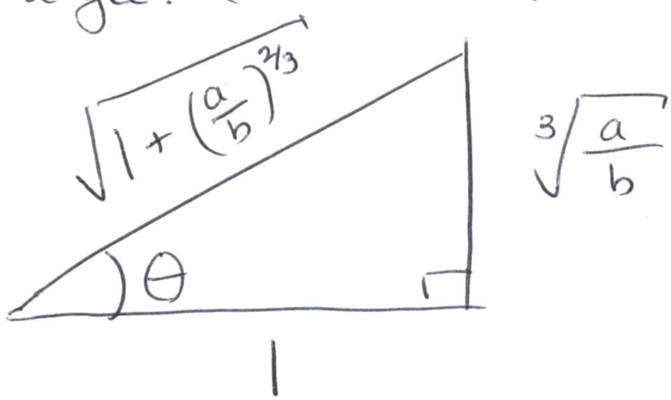
$$= \frac{\frac{\sin\theta}{\cos\theta^2}}{\frac{\cos\theta}{\sin\theta^2}} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$$

Therefore $\sqrt[3]{\frac{a}{b}} = \tan\theta \longrightarrow \theta = \tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right)$.

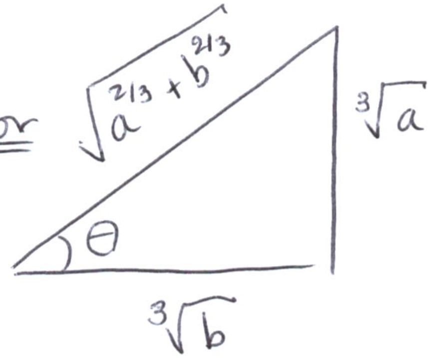
Now we check endpoints and critical values to find the min.
At the endpoints $\theta = 0$ and $\theta = \frac{\pi}{2}$, $L(\theta)$ is not defined.
(Observe $L \to \infty$ as $\theta \to 0^+$ and $\theta \to \frac{\pi}{2}^-$).
On the other hand, at $\theta = \tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right)$ we get:

$$L = \frac{a}{\sin\left(\tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right)\right)} + \frac{b}{\cos\left(\tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right)\right)}$$

From the triangle: (use $\tan\theta = \sqrt[3]{\frac{a}{b}}$ to build the triangle)



or



we get

$$\sin\left(\tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right)\right) = \frac{\sqrt[3]{a}}{\sqrt{a^{2/3}+b^{2/3}}}$$

$$\cos\left(\tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right)\right) = \frac{\sqrt[3]{b}}{\sqrt{a^{2/3}+b^{2/3}}}$$, so the length is

$$L = a^{2/3} \cdot \sqrt{a^{2/3}+b^{2/3}} + b^{2/3} \cdot \sqrt{a^{2/3}+b^{2/3}}$$

$$= \sqrt{a^{2/3}+b^{2/3}}\left(a^{2/3}+b^{2/3}\right) = \left(a^{2/3}+b^{2/3}\right)^{3/2}.$$

<u>We are not done!</u> This could, unfortunately, be a local max (and not a min as we wanted). So we must convince ourselves it is a local min. Using the second derivative, we see:

$$L'' = a\csc\theta \underbrace{\left(\csc^2(\theta) + \cot^2\theta\right)}_{\text{always pos}} + b\sec\theta \underbrace{\left(\sec^2\theta + \tan^2\theta\right)}_{\text{always pos}}$$

positive when θ is in $[0, \frac{\pi}{2}]$.

So overall, $L'' > 0$ on its domain. Thus, the point $\left(\tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right), \left(a^{2/3}+b^{2/3}\right)^{3/2}\right)$ is a local minimum by the second derivative test.

In general, the procedure is:

① Draw a picture, if appropriate.

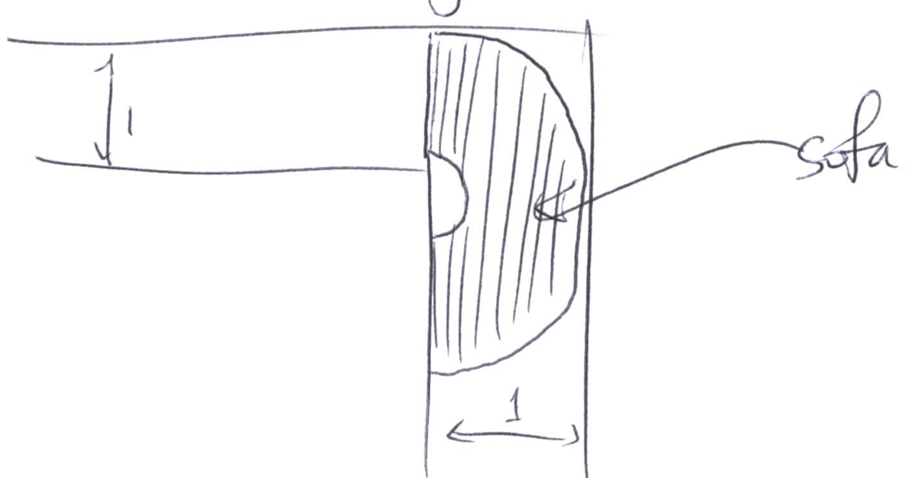
② Define quantities (label the diagram)

③ Express the quantity Q to be minimized/maximized in terms of other quantities.

④ If Q is expressed in terms of n variables, use/find $n-1$ equations to reduce Q to a function of a single variable

⑤ Determine the domain of Q, and maximize or minimize Q over that domain, in the standard way.

Remark: What we did is a special case of the "moving sofa problem", where halls are taken to be unit width and you try to find the largest object (sofa) that can go around the corner.



This is currently an open problem.