MATH 1230

§4.1 Related Rates

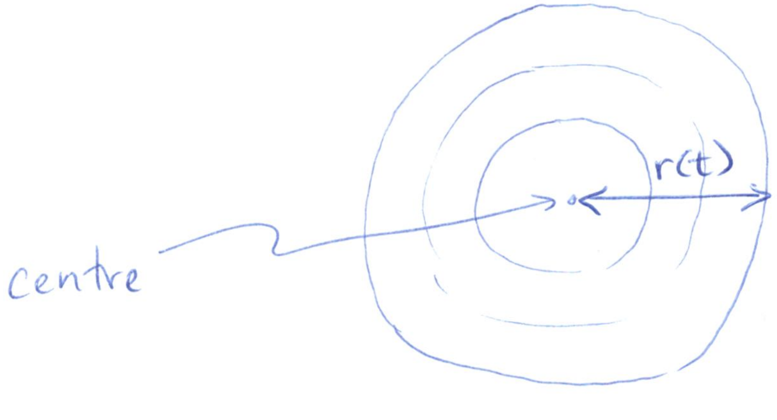
A <u>related rate problem</u> is a type of word problem centred around the following idea:

Given a relation between quantities that change with time, we can differentiate to obtain a relation involving rates of change.

<u>Example</u>: A pebble is dropped into a still pond, creating circular ripples whose radius grows at a rate of 1m/s. How fast is the area of the ripples growing when the radius is 2m?

<u>Solution</u>: We have:

Radius $r(t)$, and $\frac{dr}{dt} = 1\,m/s$. The picture is:



The area is $A = \pi r^2$ (this relates our known quantity—radius—to the unknown, area).

Differentiate (implicitly):

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \quad \text{(recall } r = r(t) \text{ is a function of } t\text{)}$$

So, if we want to know $\frac{dA}{dt}$ when $r(t) = 2m$, we can

simply plug in $r(t) = 2$, $\frac{dr}{dt} = 1 \, m/s$ and get:

$$\frac{dA}{dt} = 2\pi(2) \cdot 1 = 4\pi \, m^2/s.$$

Answer: The area is changing at $4\pi \, m^2/s$ when $r(t) = 2$.

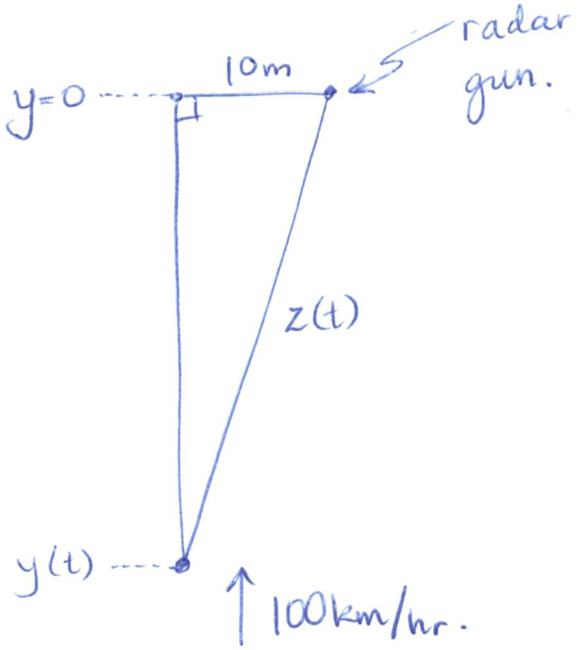
Example: Radar speed guns measure the rate of change of the distance between a moving object and the gun.

Suppose a car is traveling 100 km/hr down a straight road towards a radar gun that is placed 300m down the road, and 10m to the side. What is the radar gun's measurement of the car's speed?

Solution: We're given:

The speed of the car, and distance to the radar gun.

So we let $y(t)$ denote the position of the car along the road (so $\frac{dy}{dt} = 100$ km/hr) as in the diagram:



<u>Attention</u>: We need to use the same unit of distance for every measurement, so replace 10m with $\frac{1}{100}$ km.

The diagram gives:

$$(y(t))^2 + \left(\frac{1}{100}\right)^2 = (z(t))^2$$

at all times t.

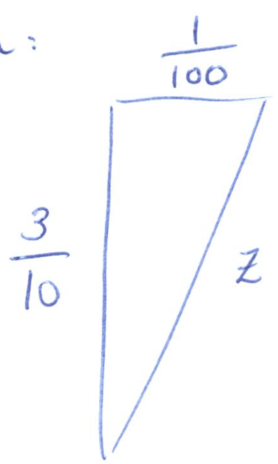
We're asked to find $\frac{dz}{dt}$ given that $\frac{dy}{dt} = -100$km/hr

and $y(t) = \frac{3}{10}$ km (300 meters).

<u>Differentiate</u>: $\cancel{2}y\frac{dy}{dt} + 0 = \cancel{2}z\frac{dz}{dt}$

$$\Rightarrow \quad y\frac{dy}{dt} = z\frac{dz}{dt}.$$

We know y, $\frac{dy}{dt}$ but <u>not</u> z when $y(t) = 300$m. We compute z from:



$$\Rightarrow z^2 = \left(\frac{3}{10}\right)^2 + \left(\frac{1}{100}\right)^2 = \frac{901}{100^2}$$

$$\Rightarrow z = \frac{\sqrt{901}}{100}$$

Now plugging in known values:

$$\frac{3}{10}(-100) = \frac{\sqrt{901}}{100}\frac{dz}{dt} \Rightarrow \frac{dz}{dt} = -99.9 \text{ km/h}.$$

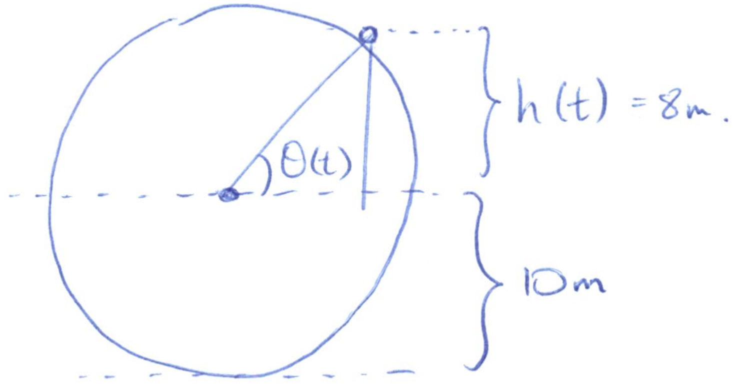
(so close, but not exact).

Example: A Ferris wheel with a radius of 10m is rotating at a rate of one revolution every 2 minutes. How fast is the rider rising when they are 18m above the ground?

Solution: We are given the rate of change of $\theta(t)$, the angle the rider forms with a parallel to the ground: $\frac{d\theta}{dt} = \frac{2\pi}{2\text{min}} = \pi/\text{min}$.

Let $h(t)$ denote the height of the rider above the axis of rotation (we're asked to find $\frac{dh}{dt}$ when $h(t) = 8$m).

Picture:

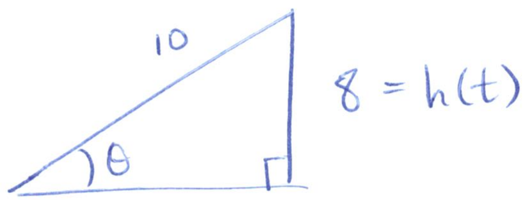


To produce an equation relating these quantities, note $\sin(\theta(t)) = \frac{\text{opp}}{\text{hyp}} = \frac{h(t)}{10}$. Differentiating:

$$\frac{d}{dt}\sin(\theta(t)) = \frac{d}{dt}\left(\frac{h(t)}{10}\right)$$

$$\Rightarrow \cos\theta(t)\,\frac{d\theta}{dt} = \frac{1}{10}\,\frac{dh}{dt}.$$

We know $\frac{d\theta}{dt}$. To determine $\cos\theta(t)$ when $h(t)=8$, observe that we have a triangle:



$\sqrt{100^2-8^2} = \sqrt{36} = 6.$

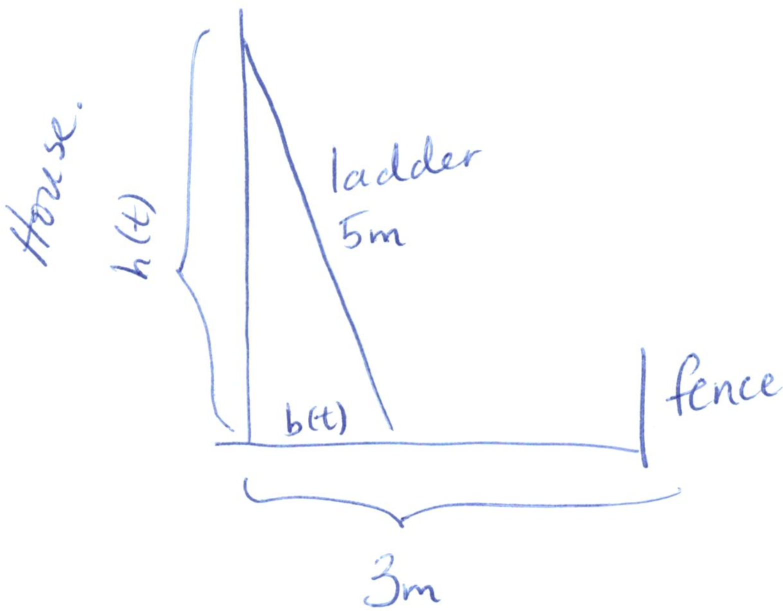
$$\Rightarrow \cos(\theta(t)) = \frac{3}{5} \text{ when } h(t)=8.$$

Therefore $\frac{3}{5}\cdot\pi = \frac{1}{10}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{30\pi}{5}$ m/min.

<u>Example</u>: A house is 3m away from a fence marking the edge of the property. A ladder 5m long is leaning against the house, and the top of the ladder begins to slide down at 1m/s. How fast is the bottom of the ladder strike the fence? (i.e. at what speed?)

<u>Solution</u>: We are given information about $h(t)$, the height of the top of the ladder. We know $\frac{dh}{dt} = -1$. Let $b(t)$ denote the distance of the foot of the ladder from the house.

Picture:



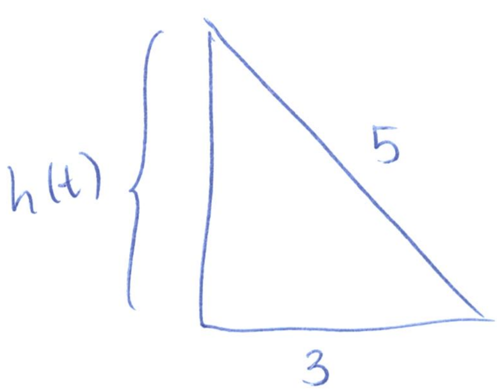
We want ~~dt~~ $\frac{db}{dt}$ when $b(t)=3$.

Then $(b(t))^2 + (h(t))^2 = 5^2$

$$\Rightarrow 2b(t)\frac{db}{dt} + 2h(t)\frac{dh}{dt} = 0$$

$$\Rightarrow \frac{db}{dt} = -h(t)\frac{dh}{dt}\cdot\frac{1}{b(t)}$$

When $b(t)=3$, we have



$$\Rightarrow h(t) = \sqrt{5^2 - 3^2} = \sqrt{16} = 4.$$

Therefore $\frac{db}{dt} = -4\cdot(-1)\cdot\frac{1}{3} = \frac{4}{3}$ m/s.

MATH 1230

§4.3 L'Hôpital's Rule (~1696, sometimes attributed to Bernoulli).

L'Hôpital's rule is a method of evaluating limits that yield problems using other methods.

E.g. $\lim_{x\to 0} \frac{\sin x}{x} = 1$, but if we simply take the limits of the top and bottom we get $\frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} x} = \frac{0}{0}$. This is called an indeterminate form of type $\frac{0}{0}$.

Similarly, $\lim_{x\to 0} \frac{\ln(1/x^2)}{\cot(x^2)}$ is an indeterminate form of type $\frac{\infty}{\infty}$, since $\frac{\lim_{x\to 0} \ln\left(\frac{1}{x^2}\right)}{\lim_{x\to 0} \cot(x^2)} = \frac{\infty}{\infty}$.

~~First~~ L'Hôpital's rule:

Suppose that f and g are both differentiable on (a,b), and that c is a number in (a,b). Assume $g'(x) \neq 0$ anywhere in (a,b), except possibly at c. Then

(i) If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or is equal to $\pm\infty$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

(ii) If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \pm\infty$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists

or is $\pm\infty$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

Remarks: ① We could also have $a=-\infty$, $b=+\infty$ in the theorem above, and replace $x\to c$ with $x\to a^+$, $x\to b^-$, or $x\to\pm\infty$.

② The book states L'Hôpital's rule incorrectly and as a result this section is replete with logical inconsistencies. A better resource for the technical matters behind L'Hôpital's rule is wikipedia (the error is that $g'(x)\neq 0$ for all (a,b))

Example: Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution: Note that $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} x = 0$,

so we have an indeterminate form $\frac{0}{0}$. Note also that $\frac{d}{dx}(x) = 1$, so $g'(x)\neq 0$ (where $g(x)$ is the bottom).

So by L'Hôpital's rule

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example: Evaluate $\displaystyle\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$ (Example 2 in the book).

Solution: First note that

$$\lim_{x\to 0} 2\sin x - \sin(2x) = 0 \quad \text{and} \quad \lim_{x\to 0} 2e^x - 2 - 2x - x^2 = 0.$$

So it's a $\frac{0}{0}$ indeterminate form. L'Hôpital's rule gives:

$$\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$$

$$= \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{2e^x - 2 - 2x} \quad \Big\} \text{ differentiate top and bottom.}$$

Now $\displaystyle\lim_{x\to 0} 2\cos x - 2\cos(2x) = 2 - 2 = 0$ and

$$\lim_{x\to 0} 2e^x - 2 - 2x = 0$$

So it's still $\frac{0}{0}$. Try again:

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin(2x)}{2e^x - 2} \quad \Big\} \text{ diff top and bottom}$$

We still get $\frac{0}{0}$. Try again:

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos(2x)}{2e^x}$$

$$= \frac{-2+8}{2} = 3.$$

Example: Evaluate $\lim\limits_{x\to\infty} \frac{x^n}{e^x}$.

Solution: Since $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to\infty} e^x = \infty$, this is an indeterminate form $\frac{\infty}{\infty}$. So we apply L'Hôpital's rule:

$$\left.\begin{aligned} \lim_{x\to\infty} \frac{x^n}{e^x} &\overset{HR}{=} \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} \quad \left(\text{still } \tfrac{\infty}{\infty}\right) \\ &\overset{HR}{=} \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \quad \left(\text{still } \tfrac{\infty}{\infty}\right) \\ &\overset{HR}{=} \lim_{x\to\infty} \frac{n(n-1)(n-2)x^{n-3}}{e^x} \quad \left(\text{still } \tfrac{\infty}{\infty}\right) \end{aligned}\right\} \text{as long as } n-1,\ n-2 \text{ and } n-3 \text{ are not } 0!$$

$\vdots$ keep going until $x^{n-n} = x^0 = 1$, we get

$$= \lim_{x\to\infty} \frac{n!}{e^x} = n! \lim_{x\to\infty} e^{-x} = 0, \text{ done.}$$

Remark: Though the intermediate steps will be messier, this is true of any polynomial/exponential quotient. That is,

$$\lim_{x\to\infty} \frac{\text{polynomial}}{\text{exponential}} = 0,$$

as long as the exponential has base bigger than 1 (ie. a^x for $a>1$, so that it grows as $x\to\infty$ instead of shrinking).

There are many other indeterminate forms, like $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, etc... generally one deals with these by trying to algebraically change them into $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

<u>Example</u>: Evaluate $\lim_{x\to\infty}\left(1+\sin\left(\frac{3}{x}\right)\right)^x$.

<u>Solution</u>: This is of the form 1^∞, since $\lim_{x\to\infty} 1+\sin\left(\frac{3}{x}\right)=1$ and $\lim_{x\to\infty} x=\infty$. However we can change it to $\frac{0}{0}$.

If $L=\lim_{x\to\infty}\left(1+\sin\left(\frac{3}{x}\right)\right)^x$, then

$$\ln(L)=\ln\left(\lim_{x\to\infty}\left(1+\sin\left(\frac{3}{x}\right)\right)^x\right)$$

$$=\lim_{x\to\infty}\ln\left(1+\sin\left(\frac{3}{x}\right)\right)^x$$

limit laws: can pass a limit inside a continuous function.

$$=\lim_{x\to\infty} x\ln\left(1+\sin\left(\frac{3}{x}\right)\right)$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\sin\left(\frac{3}{x}\right)\right)}{\frac{1}{x}}$$

algebra trick!

Now $\lim_{x\to\infty}\ln\left(1+\sin\left(\frac{3}{x}\right)\right)=\ln(1)=0$ and $\lim_{x\to\infty}\frac{1}{x}=0$.

So it's $\frac{0}{0}$, and we can apply L'Hopital's rule:

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\sin\left(\frac{3}{x}\right)}\cdot\cos\left(\frac{3}{x}\right)\cdot\frac{-3}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}}$$

$$= \lim_{x\to\infty} \frac{3\cos\left(\frac{3}{x}\right)}{1+\sin\left(\frac{3}{x}\right)} = \frac{3}{1} = 3.$$

Finally, a word of caution: If you attempt to apply L'Hôpital's rule to a limit that is <u>not</u> an indeterminate form, you may get the wrong answer.

<u>Example</u> (<u>WRONG</u> method)!

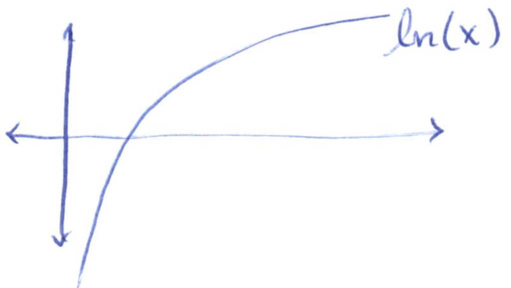
Evaluate $\lim_{x\to 1^+} \frac{x}{\ln(x)}$.

<u>Solution</u>: (WRONG method first)!

Suppose we tried L'Hôpital's rule. Then we'd say:

$$\lim_{x\to 1^+} \frac{x}{\ln(x)} = \lim_{x\to 1^+} \frac{1}{\frac{1}{x}} = \lim_{x\to 1^+} x = 1.$$

However this is <u>wrong</u>! Recall that $\ln(1)=0$, and if $x>1$ then $\ln(x)>0$ (the graph is



)

So in fact $\lim_{x\to 1^+} \frac{x}{\ln(x)} = \infty$, since the bottom is positively approaching 0 as the top approaches 1.

MATH 1230

§4.4 Extreme Values.

If $f(x)$ is a function, then the max/min values of $f(x)$ together are referred to as extreme values. There are two kinds of extreme values: absolute and local.

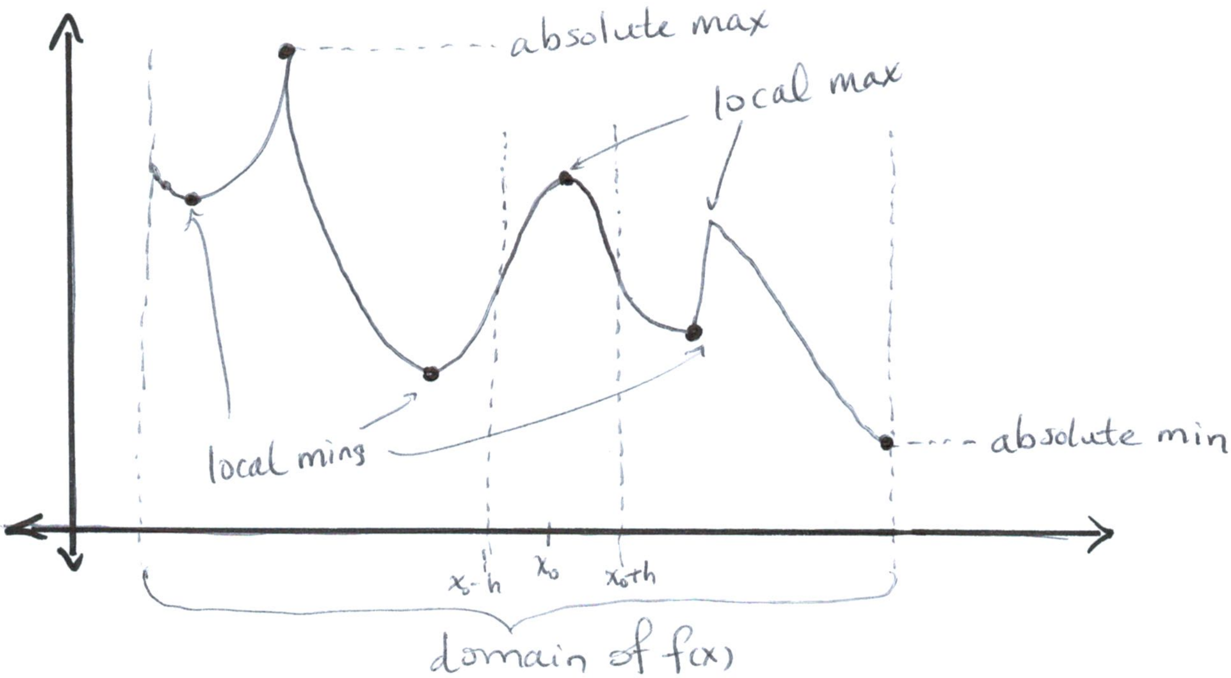
Absolute: A point x_0 in the domain of $f(x)$ gives an absolute max value $f(x_0)$ of $f(x)$ if $f(x) \le f(x_0)$ for all x in the domain.

Similarly $f(x)$ an absolute min $\iff f(x) \ge f(x_0)$ for all x in the domain.

Local: A point x_0 gives a local max value of $f(x_0)$ if there's an $h>0$ such that $f(x) \le f(x_0)$ whenever $|x-x_0|<h$.

Similarly $f(x_0)$ a local min $\iff f(x) \ge f(x_0)$ whenever $|x-x_0|<h$.

Pictures:



We would like to be able to find all x_0 in the domain of a given function $f(x)$ such that $f(x_0)$ is an extreme value. We would then also like to be able to say if $f(x_0)$ is a max or a min.

It turns out that if $f(x_0)$ is an extreme value, then x_0 is one of three kinds of points (if $f(x)$ is continuous):

(i) A <u>critical point</u> of $f(x)$, ie a point where $f'(x)=0$

(ii) An <u>singular point</u> of $f(x)$, ie. a point where $f'(x)$ is not defined, or

(iii) An <u>endpoint</u> of the domain of $f(x)$.

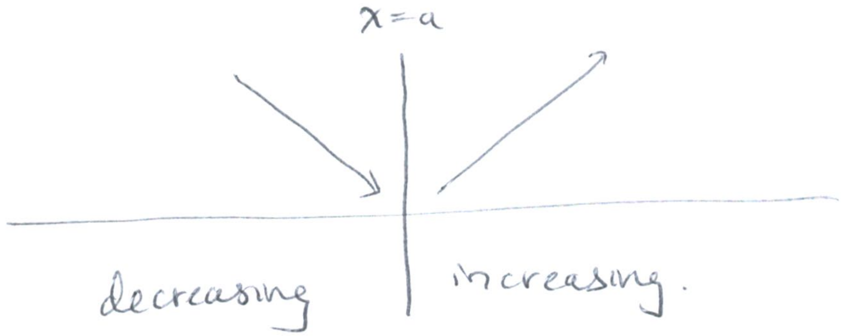
[This follows from work we already did, recall we showed that $f'(x)>0 \Rightarrow$ increasing, $f'(x)<0 \Rightarrow$ decreasing, and $f'(x)=0$ must therefore be a max, or min, or possibly a "flat spot"].

So, <u>here is how</u> we find extreme values; and classify them:

① Calculate $f'(x)$, and list all points in the domain of $f(x)$ where $f'(x)=0$ or $f'(x)$ is undefined.

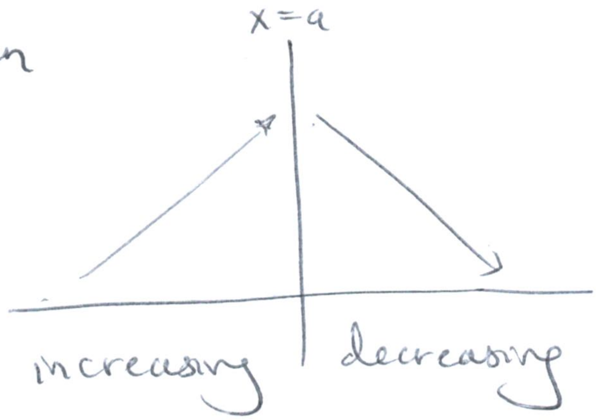
② For each point in the list from step 1, we check if it's a max or min by looking at the sign of $f'(x)$.

If $x=a$ is in the list and $f'(x)<0$ left of a, $f'(x)>0$ right of a, then $f(x)$ looks like:



So $x=a$ is obviously a minimum.

If $x=a$ is in the list and $f'(x)>0$ for $x<a$, $f'(x)<0$ for $x>a$, then



and $x=a$ is a max.

If $f(x)$ does not change sign at $x=a$, then we don't have a max or min.

Example: Find and classify the extreme values of

$$f(x) = 2x - \sin^{-1}(x).$$

Solution: First, we note that the domain of $f(x)$ is $[-1, 1]$ since the domain of $\sin^{-1}(x)$ is $[-1, 1]$. Next, we compute

$$f'(x) = 2 - \frac{1}{\sqrt{1-x^2}}.$$

Now we calculate: $f'(x) = 0 \Rightarrow \frac{1}{\sqrt{1-x^2}} = 2$

$$\Rightarrow \sqrt{1-x^2} = \frac{1}{2}$$

$$1 - x^2 = \frac{1}{4}$$

$$\Rightarrow x = \frac{\pm\sqrt{3}}{2}, \text{ note } \frac{\pm\sqrt{3}}{2} \text{ is in } [-1, 1]$$

Also note that $f'(x)$ is only undefined at the endpoints $x = \pm 1$ of the domain, because there we get division by zero.

So our list is: $\{-1, \frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1\}$.

Next we need to determine the sign of $f'(x)$ on each of the intervals $[-1, -\frac{\sqrt{3}}{2}]$, $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$, $(\frac{\sqrt{3}}{2}, 1)$ in order to classify the points from the previous step. ~~(Actually, the sign of $f'(x)$ changes on $(\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$.)~~ We can choose 'test points' for $f'(x)$ in each interval:

In $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ we test $x=0$ and find

$$f'(0) = 2 - \frac{1}{\sqrt{1-0}} = 1, \text{ so } f'(x) \text{ is positive on } \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right).$$

On $\left(-1, -\frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, 1\right)$ we test at $\pm \frac{8}{7} \cdot \frac{\sqrt{3}}{2} = \frac{8\sqrt{3}}{14}$.

$$f'\left(\frac{\pm 8\sqrt{3}}{14}\right) = 2 - \frac{1}{\sqrt{1 - \frac{192}{196}}} = 2 - \frac{1}{\sqrt{\frac{1}{49}}} = 2 - \frac{1}{\frac{1}{7}} = \underline{\underline{-5}}.$$

So $f'(x)$ is negative on $\left(-1, -\frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, 1\right)$. Thus we have:

		$x = -\frac{\sqrt{3}}{2}$		$x = +\frac{\sqrt{3}}{2}$	
$f'(x)$	$-$		$+$		$-$
$f(x)$	↘ decr.		↗ incr.		↘ decr.

Therefore $x=1$ and $x = -\frac{\sqrt{3}}{2}$ are <u>minima</u>, and

$x=-1$ and $x = \frac{\sqrt{3}}{2}$ are <u>maxima</u>.

"<u>By the first derivative test</u>" (Say this to explain how you arrived at your answer).

To determine which is an absolute max and which is a local max, we plug in:
(or min)

$$f(1) = 2(1) - \sin^{-1}(1)$$

what angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ gives $\sin(\theta) = 1$?

$$= 2 - \frac{\pi}{2} \simeq 0.43$$

and

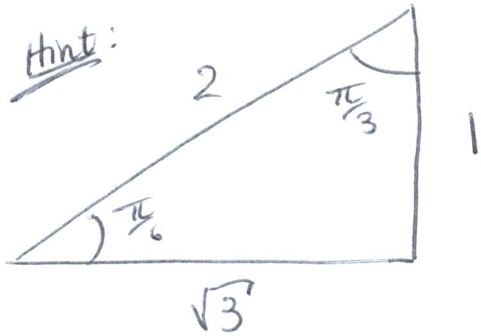
$$f\left(\frac{-\sqrt{3}}{2}\right) = 2\left(\frac{-\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

what angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ gives $\sin(\theta) = -\frac{\sqrt{3}}{2}$?

$$= -\sqrt{3} - \left(-\frac{\pi}{3}\right)$$

$$\simeq -0.68.$$

Hint:

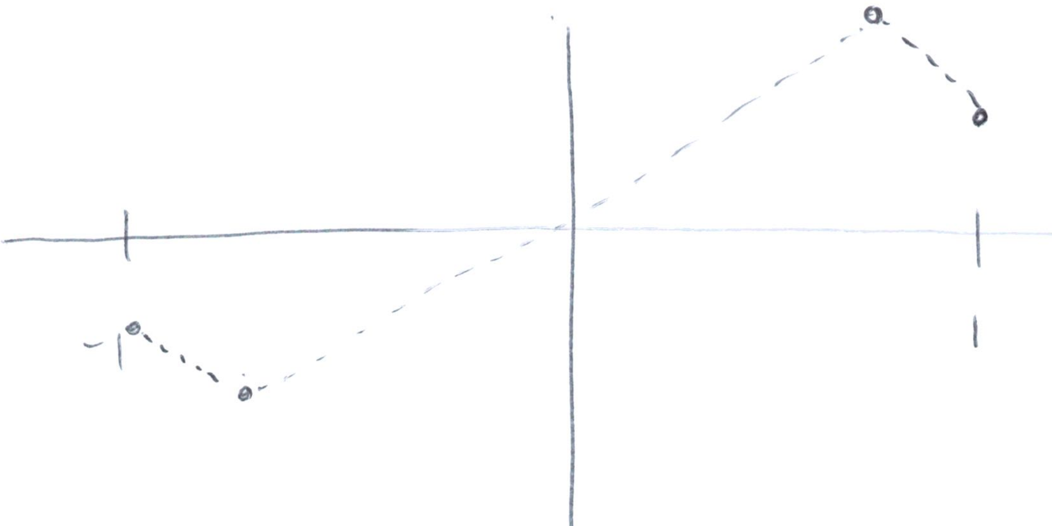


So $x_0 = \frac{-\sqrt{3}}{2}$ gives the absolute min of

$$f\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{3} + \frac{\pi}{3}.$$

Similarly (or, by symmetry) $f\left(\frac{\sqrt{3}}{2}\right)$ is the absolute max value of $f(x)$.

So the graph of $f(x)$ looks something like:



However, the shape of the graph is a mystery where we have drawn dotted lines. Next class we learn how to analyze these unknown parts.