Quick Introduction to symplectic geometry

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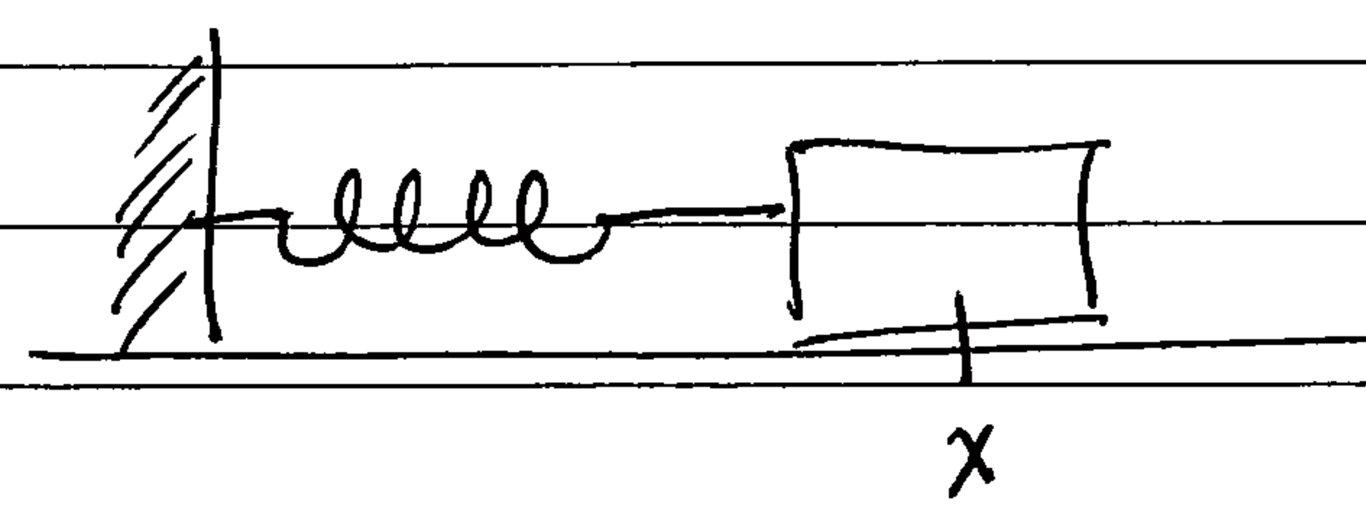
This will be (more or less) an introduction to basic definitions in preparation for Dinamo and a postdoc talk.

<u>Example</u>: Hooke's Law for springs.



$x''(t) = -k\,x(t)$, k = spring constant.
Change variables $q = x$, $p = x'$, then we get

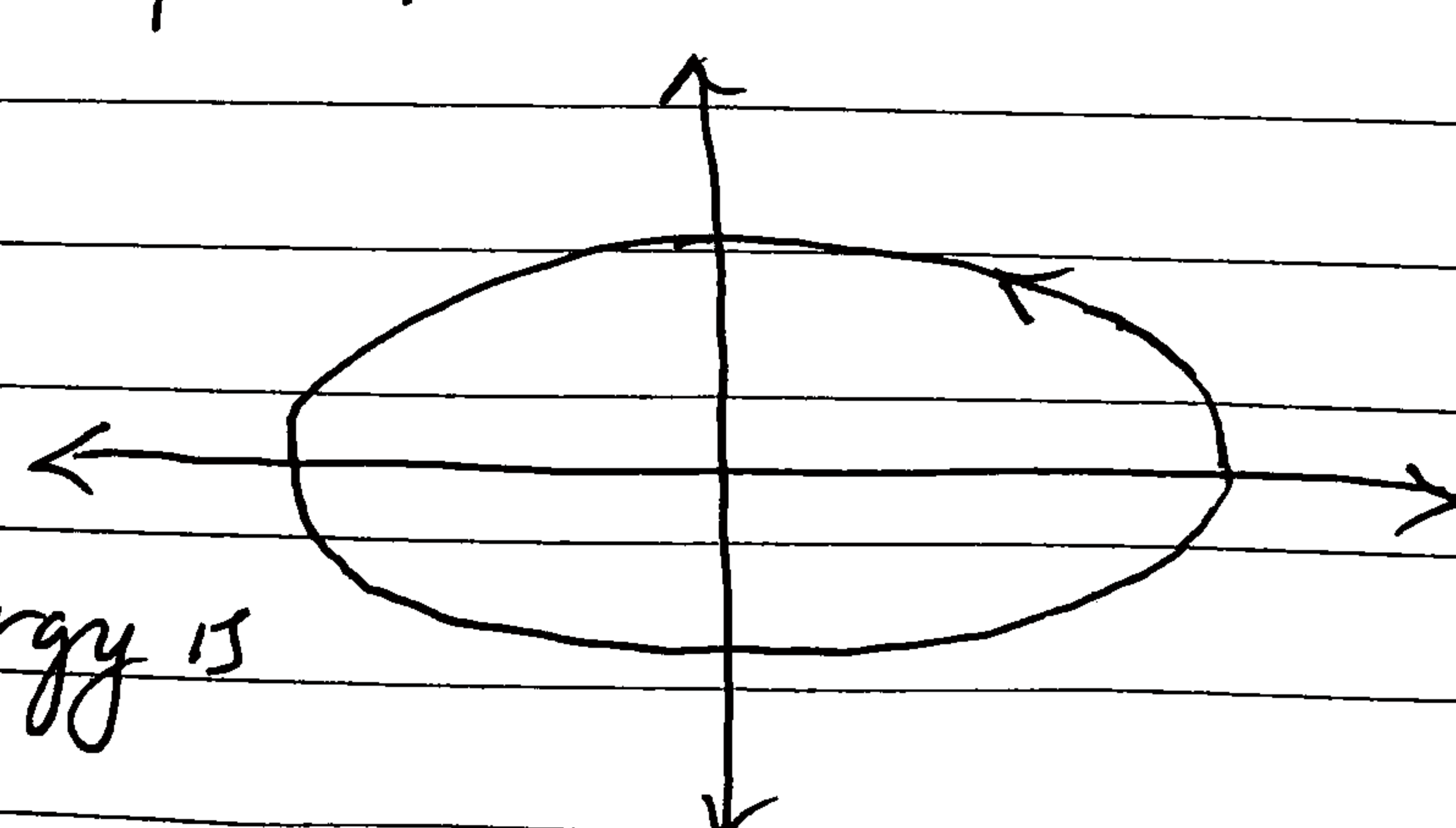
$$q'(t) = p(t) \quad \text{and} \quad p'(t) = q(t),$$

a pair of coupled first order ~~syst~~ eqns.

We find a solution trajectory that's elliptical:

$$q = q_0 \cos(\sqrt{k}\,t) \qquad p = -q_0\sqrt{k}\,\sin(\sqrt{k}\,t)$$

$$\Rightarrow \; kq^2 + p^2 = kq_0.$$



Can observe: total energy is

$$H(q,p) = \text{kinetic} + \text{potential}$$
$$= \frac{kq^2}{2} + \frac{1}{2}p^2 \quad \text{(assuming mass 1)}$$

Then we observe that along the trajectory, $H(p,q)$ is constant.

Alternatively, one can start with the total energy function – the <u>Hamiltonian</u> $H(p,q)$, and we try to find solutions that evolve along curves of constant energy, i.e. level sets of the Hamiltonian.

In this special case, we observe that the gradient ∇H gives a normal vector to the level sets, so we want ∇H rotated by ~~90?~~ $90°$ in order to get a vector field whose flow lines are solutions. I.e, we want

$$(q', p') = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$$ "skew gradient". "Hamilton's eqns of motion"

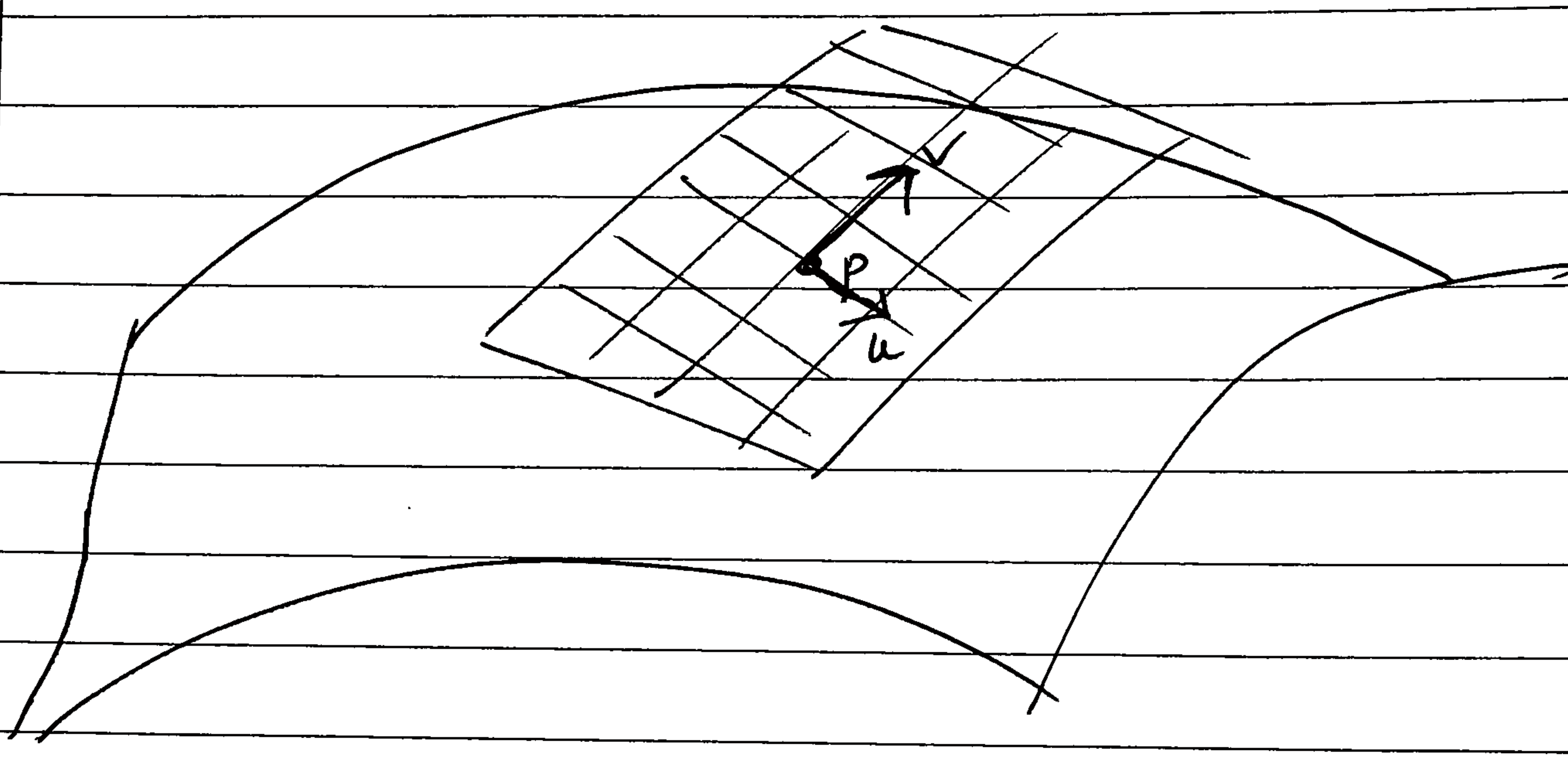
<u>General setup</u>:

Given a smooth manifold M, we think of M as the "phase space" of possible position/momenta

+ we need a closed, non-degenerate differential 2-form which encodes how we can write down Hamilton's equations.

How to think of the 2-form $\omega \in \Omega^2(M)$?

Here's a piece of M:



At each $p \in M$, ω gives $\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$
a ~~form~~ antisymmetric bilinear form.

E.g.

$$(u,v) \longmapsto \begin{bmatrix} - & u & - \end{bmatrix} \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right] \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

$2n \times 2n$, so even dimension is required of u, v, M.

and locally, (in suitable coordinates) ω_p is exactly of this form.

Note that non-degenerate is automatic, and even dimension is forced.

What does the non-degeneracy do for us?

Non-degeneracy yields a "skew gradient" of functions H, called X_H. It satisfies

$$\omega(X_H, v) = dH(v). \qquad (*).$$

The RHS is the directional derivative of H at v, a number for each v.

Non-degeneracy gives a unique solution of $(*)$, ie. a vector field X_H that happens to vary smoothly (need to check this).

Now the "skew-gradient" is not the usual terminology, it's typically called the Hamiltonian vector field of H.

Then the Hamiltonian vector field allows for Hamilton's equations $\frac{d}{dt}x(t) = X_H(x(t))$

Does this yield the property that: The function H is constant along any solution trajectory?

Here we use antisymmetry.

$$dH(x'(t)) = dH(X_H) = \omega(X_H, X_H) = 0$$

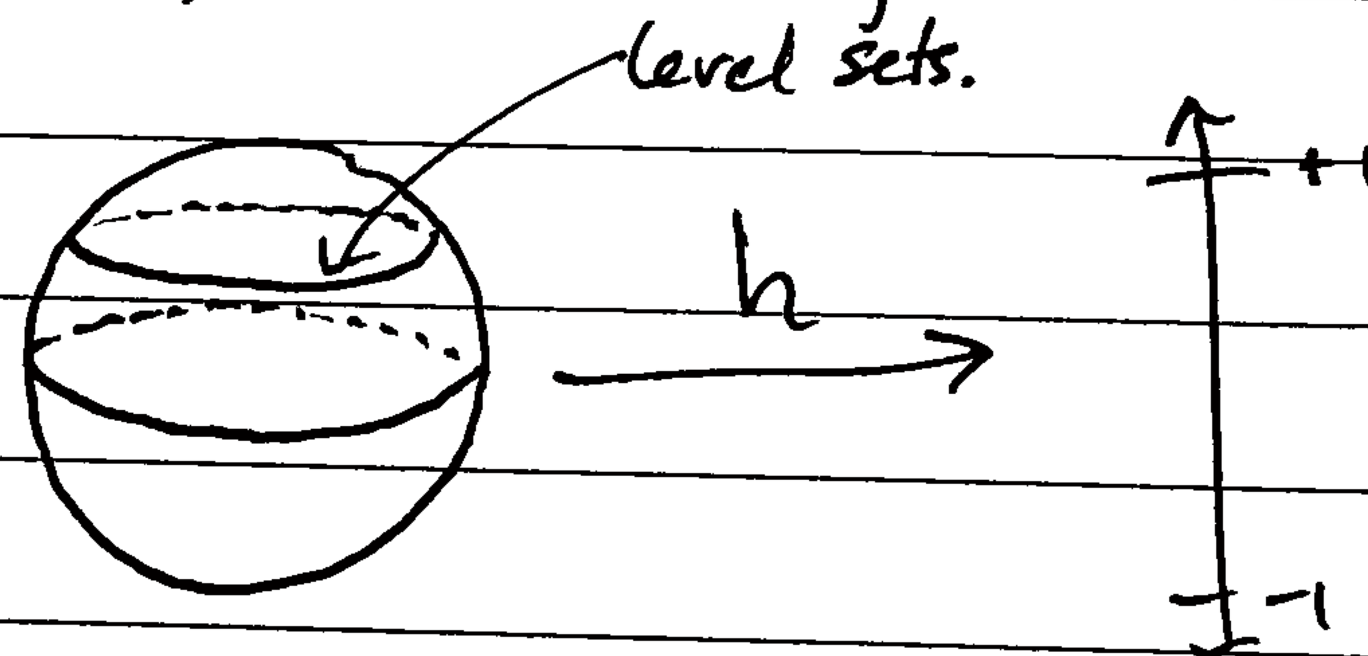
using skew symmetry.

$\Rightarrow H$ is constant along solution trajectories to Hamilton's equations, as desired.

Example ① $M = \mathbb{R}^{2n}$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

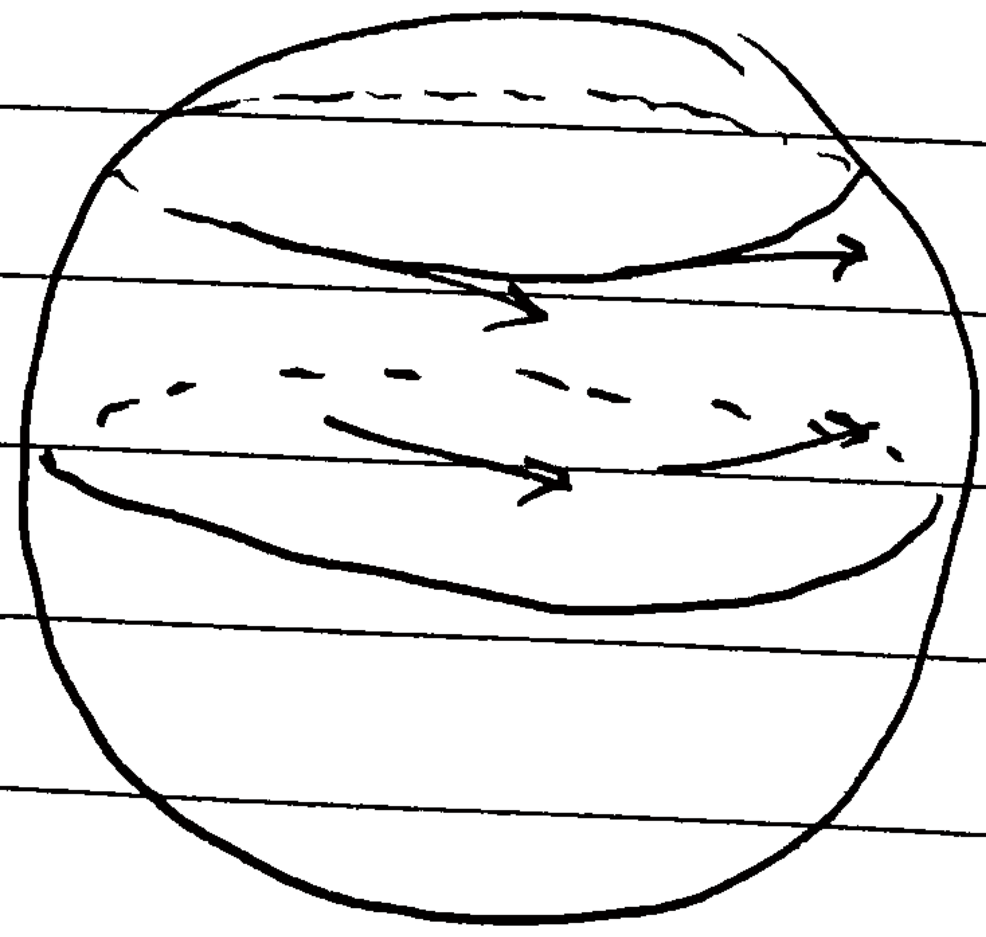
Here the x_i's are the first n coords, y_i the last.

② $M = S^2$, $\omega =$ area form



(the height function)

Then the latitude lines are the level sets of the Hamiltonian h, the Hamiltonian vector field is just



Remark: The 2-sphere is the only sphere that admits a symplectic ω.

③ M oriented surface also admits ω

④ $M = T^*X$, X any manifold (cotangent)

this always admits such an ω.

⑤ coadjoint orbits in the dual of a Lie algebra.

⑥ $\mathbb{C}P^n$, or more generally toric varieties.

Poisson algebra (of observables)

We can define

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

where $C^\infty(M)$ is all $M \to \mathbb{R}$. We can do this:

$$\{f,g\} = \omega(X_f, X_g),$$ ie the value of $\{f,g\}$ at p is the X_f, X_g at p plugged into ω.

This guy $\{f,g\}$ has properties:

- bilinear, skew-symmetric, and
- $\{\cdot, g\} = X_g$ so it satisfies a Leibniz rule

$$\{f_1 f_2, g\} = \{f_1, g\} f_2 + f_1 \{f_2, g\}$$

and the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \underbrace{\{h,\{f,g\}\}}_{} = 0.$$ $\mathrm{Jac}(f,g,h)$

What does the Jacobi identity mean?

Lemma $\mathrm{Jac}(f,g,h)$

$$= X_{\{f,g\}}(h) - [X_g, X_f](h)$$

here, $[X,Y] = XY - YX$

In particular, $\mathrm{Jac}(f,g,h) = 0$ exactly when $X_{\{f,g\}} = [X_g, X_f]$ ie we've got

$$\begin{aligned} C^\infty(M) &\longmapsto \mathfrak{X}(M) \\ f &\longmapsto X_f \\ \{f,g\} &\longmapsto -[X_f, X_g] \end{aligned}$$