Generalized torsions in amalgam and 3-manifold groups

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Abstract:

A group is left-orderable (bi-orderable) if there is a total order invariant under multiplication from the LHS (resp. both sides). A group is generalized torsion free if no product of conjugates of any nontrivial element is trivial. We will recall a sufficient condition ensuring an amalgam of two groups to be generalized torsion free. As applications, we prove two results: (1) there is a 3-manifold whose fundamental group is generalized torsion free and not orderable and (2) there is a group which is generalized torsion free and not left orderable. These results resolve a conjecture of Motegi and Teragaito and Problem 16.48 of the Kourovka Notebook (which is also Question 2.1 in Unsolved problems in ordered and orderable groups, arXiv:0906.2621), respectively. In this talk, we will focus on the proof of the first result.

This is a joint work with Adam Clay.

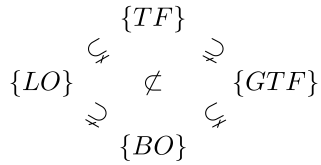

Figure 1: Containment Relations

1 Introduction and definitions

A group G is left-orderable is there is a total order $<$ on G which is invariant under multiplication from LHS; i.e., $a < b$ implies $ca < cb$ for all $a, b, c \in G$. We call such an order a left-ordering of G. Similarly, we say G is bi-orderable if there is a total $<$ order on G invariant under multiplication from both sides and call this $<$ a bi-ordering on G.

An element g in a group G is called a generalized torsion element if there are $h_1, \cdots, h_n \in G$ with $n \geq 1$ such that $g^{h_1} \cdots g^{h_n} = 1$. Note that a torsion element (a nontrivial element of finite order) is a generalized torsion element. We say a group is generalized torsion free (resp. torsion free) if it doesn't contain a generalized torsion (resp. torsion) element.

We write LO, BO, TF, and GTF for the words left-orderable, bi-orderable, torsion free and generalized torsion free respectively. We use $\{LO\}$ (resp. $\{BO\}$, $\{TF\}$, $\{GTF\}$) for the set of LO (resp. BO, TF, GTF) groups. The previously known containment relations among these four sets are given by Figure 1.

The only unknown relation is whether $\{GTF\}$ is contained in $\{LO\}$; i.e., whether every GTF group is an LO group. This is Problem 16.48 of the well-known Kourovka Notebook and it's also Question 2.1 in Unsolved problems in ordered and orderable groups, arXiv:0906.2621. As it's shown in Figure 1, it's known that a GTF group might not be bi-orderable. However, it's not known whether this is still true if we only consider a special type of group. For instance, it's conjectured by Motegi and Teragaito that for the fundamental groups of 3-manifolds, one is GTF if and only it's BO. Now we are able to solve these two problems; we prove that there is a group which is GTF and not LO and there is a 3-manifold whose fundamental group is GTF and not BO. The proof is based on a theorem which provide a sufficient condition for an amalgam of two groups to be GTF. (We have reported this result in this seminar last year and we will recall it in this talk.)

In this talk, we focus on the construction of the GTF and non-BO 3-manifold group. We will first give the construction of the 3-manifold, recall the aforementioned GTF-amalgam theorem, and then briefly discuss the proof that the 3-manifold group is non-BO and GTF.

2 The construction of the 3-manifold

First, we let K be the figure-8 knot, as shown in Figure 2. Let $N(K)$ be a closed tubular neighborhood of K and let M be complement of interior of $N(K)$ in $\mathbb{R}^3$. Then M is a closed 3-manifold. Let M_1 and M_2 be two copies of M. The boundaries ∂M_1 and ∂M_2 are homeomorphic to the torus S^2, thus have fundamental groups isomorphic to $\mathbb{Z}^2$. Let μ_i, λ_i be the generator of the abelian group $\pi_1(\partial M_i)$. We will construct an homeomorphism $\phi : \partial M_1 \to \partial M_2$ such that the induced isomorphism $\phi^* : \pi_1(\partial M_1) \to \pi_1(\partial M_2)$ sends μ_1 to μ_2 and λ_1 to $\mu_2\lambda_2$. We then let W be the 3-manifold patching M_1 and M_2 together through ϕ; i.e., $W = M_1 \sqcup_\phi M_2$ is the disjoint union of M_1 and M_2 modulo the identification $p = \phi(p)$ ($p \in \partial(M_1)$). Then by Seifert–Van Kampen theorem, we have

$$\pi_1(W) = \pi_1(M_1) *_{\phi^*} \pi_1(M_2).$$

We now write down the group $\pi(M_i)$ and the map ϕ algebraically. Since M_i are just the copies of M,

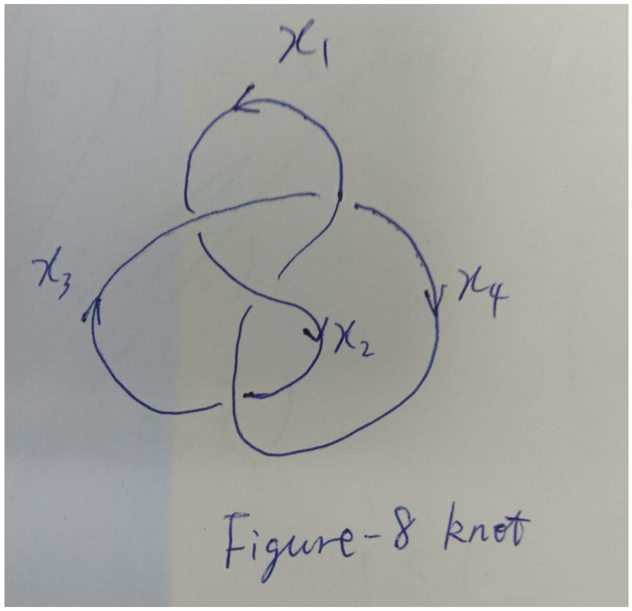

Figure 2: Figure-8 Knot

we will just consider $\pi_1(M)$. First, the group $\pi_1(M)$ is generated by x_1, x_2, x_3, x_4 subject to the following Wirtinger relations

$$x_1^{x_3^{-1}} = x_2, \quad x_2^{x_4} = x_3, \quad x_3^{x_1^{-1}} = x_4, \quad x_4^{x_2} = x_1.$$

Here we use a^b for the conjugate $b^{-1}ab$. Writing $x = x_1, y = x_2$, we have the following presentation for the group

$$\pi_1(M) = \langle x, y | wx = yw \rangle, \text{ with } w = xy^{-1}x^{-1}y.$$

As subgroup of $\pi_1(M)$, $\pi_1(\partial M)$ has meridian $\mu := x_1 = x$ and longitude

$$\lambda := x_3^{-1}x_4x_1^{-1}x_2 = (w^{-1})^{y^{-1}x}w.$$

(Note that $x_1^{x_3^{-1}x_4x_1^{-1}x_2} = x_1$ using the Wirtinger relations, thus we see algebraically that μ and λ do commute.)

Now we see how to construct the homeomorphism ϕ. Let Loop(μ) and Loop(λ) be loops in ∂M representing the meridian μ and longitude λ, respectively. There is an homeomorphism h_1 from ∂M to torus $S^2 := S^1 \times S^1$ such that it sends Loop(μ) and Loop(λ) to

$$S^1 \times \{1\} = \{(e^{2\pi is}, 1) | s \in \mathbb{R}\} \text{ and } \{1\} \times S^1 = \{(1, e^{2\pi it}) | t \in \mathbb{R}\},$$

respectively. There is a homeomorphism h_2 from S^2 to $\mathbb{R}^2/\mathbb{Z}^2$, sending $(e^{2\pi is}, e^{2\pi it})$ to $(\bar{s}, \bar{t})$ $(s, t \in \mathbb{R})$. Now, consider the homeomorphism h_3 on $\mathbb{R}^2/\mathbb{Z}^2$ by the map

$$\begin{bmatrix} \bar{s} \\ \bar{t} \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{s} \\ \bar{t} \end{bmatrix} = \begin{bmatrix} \bar{s}+\bar{t} \\ \bar{t} \end{bmatrix}.$$

Note that this homeomorphism fixes $\begin{bmatrix} \bar{s} \\ 0 \end{bmatrix}$ and send $\begin{bmatrix} 0 \\ \bar{t} \end{bmatrix}$ to $\begin{bmatrix} \bar{t} \\ \bar{t} \end{bmatrix}$. Also the loop defined by $t \mapsto (e^{2\pi it}, e^{2\pi it})$ is homotopic to the composition of the loops defined by $s \mapsto (e^{2\pi is}, 1)$ and $t \mapsto (1, e^{2\pi it})$ respectively. (We can construct the explicit homotopy.) Now, through the identification $h_2 \circ h_1$, we can use h_3 to define a homeomorphism on ∂M such that the induced isomorphism on $\pi_1(\partial M)$ sends μ to μ and λ to $\mu\lambda$. Inducing this to the copies, there is a homeomorphism ϕ from ∂M_1 to ∂M_2, such that

$$\phi^* : \mu_1 \mapsto \mu_2, \lambda_1 \mapsto \mu_2\lambda_2.$$

To wrap it up, we have, for the manifold $W = M_1 \sqcup_\phi M_2$,

$$\begin{aligned}\pi_1(W) &= \pi_1(M_1) *_{\phi^*} \pi_1(M_2)\\ &= \langle x_1, y_1 | w_1x_1 = y_1w_1\rangle *_{\phi^*} \langle x_2, y_2 | w_2x_2 = y_2w_2\rangle,\\ &= \langle x_1, y_1, x_2, y_2 | w_1x_1 = y_1w_1, w_2x_2 = y_2w_2, \mu_1 = \mu_2, \lambda_1 = \mu_2\lambda_2\rangle,\end{aligned}$$

where $w_i = x_iy_i^{-1}x_i^{-1}y_i$, $\mu_i = x_i$ and $\lambda_i = (w_i^{-1})^{y_i^{-1}x_i}w_i$.

3 Proof the $\pi_1(W)$ is non-BO and GTF

It's quite easy to prove that $\pi_1(W)$ is Non-BO. The key is that if a finitely generated group is BO, then a quotient of it is the group of integers. However, it's quite direct to verify that the abilianization of $\pi_1(W)$ is trivial, hence it can't be BO.

The proof that $\pi_1(W)$ is GTF is based on the next result:

Theorem 3.1 (C, Clay)**.** *Let $G = A *_{\phi^*} B$ be an amalgam of two groups A and B, where $\phi^* : C \to D$ is an isomorphism from a subgroup C of A to a subgroup D of B. Then G is GTF if C and D are RTF in A and B respectively and there is a family P_i $i \in I$ (resp. Q_j $j \in J$) of normal subsemigroups of A (resp. B) such that the next two conditions are satisfied*

(1) $A\backslash\{1\} = \cup_{i\in I} P_i$ and $B\backslash\{1\} \cup_{j\in J} Q_j$. (Covering condition)

(2) For every P_i (resp. Q_j) there is a Q_j (resp. P_i) such that $\phi^*(P_i \cap C) = Q_j \cap D$. (Matching condition)

A subgroup C is RTF in group A if for all $a \in A\backslash C$ and $c_1 \cdots c_n \in C$ with $n \geq 1$, $ac_1 \cdots ac_n \notin C$.

We apply this theorem to the amalgam $\pi_1(W) = \pi_1(M_1) *_{\phi^*} \pi_1(M_2)$.

(A) First, we prove that $\pi_1(M)$ is BO. Recall that $\pi_1(M) = \langle x, y | xw = wy\rangle$ with $w = xy^{-1}x^{-1}y$. Let $G = \langle a, b, t | tat^{-1} = aba, tbt^{-1} = ab$. It can be proved that the map $f : G \to \pi_1(M)$ given by $f(a) = w$, $f(b) = [w, x^{-1}]$ (where $[g, h]$ is the commutator $ghg^{-1}h^{-1}$) and $f(t) = x$ is an isomorphism. Then it follows that the subgroup H of $\pi_1(M)$ generated by w and $[w, x^{-1}]$ is normal, isomorphic to the free group F_2 of rank two. The abelianization of the conjugation by x on H, given by the matrix $\left[\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right]$, have positive real eigenvalues. Therefore, by a result of Perron-Rolfsen, there is a bi-ordering on H which is invariant under the conjugation by x. Thus, this bi-ordering can be extended to a bi-ordering of $\pi_1(M)$. Put it in another way, we can choose a positive cone Q of H such that $\lambda \in Q$ and let $R = \{g \in \pi_1(M) | f(g) > 0\}$, where $f : \pi_1(M) \to \mathbb{Z}$ is the abelianization sending x to 1, then we have $Q \cup R$ is a positive cone of $\pi_1(M)$

(B) Second, the proof that $\pi_1(\partial M)$ is RTF in $\pi_1(M)$ is based on a 1976's result of Jonathan Simon, which ensures that $\pi_1(\partial M)$ is isolated in $\pi_1(M)$. Then we use the fact that an isolated abelian subgroup of a bi-orderable group is RTF.

(C) Third, the construction of the families of the normal subsemigroups. This is the most difficult part of the proof. Let's first write down the two families as following:

$$\begin{array}{cccc} R_1^{\pm1}, & Q_1^{\pm1}, & X_{(-1,1)}^{\pm1}, & Y_1^{\pm1}\\ X_{(1,1)}^{\mp1}, & Y_2^{\pm1}, & R_2^{\pm1}, & Q_2^{\pm1}\end{array}$$

We will need to define the subsemigroups on the first (resp. second) line, which gives eight normal subgroups of $\pi_1(M_1)$ (resp. $\pi_1(M_2)$). We will see that their unions are exactly the non-trivial elements of $\pi_1(M_1)$ and $\pi_1(M_2)$, respectively. Finally, we need to show that they match each pair of subgroups in a column "matches"; meaning, for example $\phi^*(R_1^{\pm1} \cap \pi_1(\partial M_1)) = X_{(1,1)}^{\mp1} \cap \pi_1(\partial M_2)$. We will not give much detail, but make the following remarks to finish the talk.

(a) The subsemigroups R_i, Q_i are the copies in $\pi_1(M_i)$ of Q, R given in (A). Mainly because of this, the covering condition in Theorem 3.1 is satisfied.

(b) The construction of $X_{(-1,1)}$ and $X_{(1,1)}$ uses a result of Steve Boyer, Cameron Gordon, and Ying Hu, which gives some homomorphisms from $\pi_1(M)$ to the covering group of $\mathrm{Homeo}_+(S^1)$, which can be written as

$$Co := \{f \in \mathrm{Homeo}_+(\mathbb{R}) | f(x+1) = f(x) + 1, \forall x \in \mathbb{R}\}.$$

It also uses the tool of translation numbers, which gives a map from Co to $\mathbb{Z}$.

(c) The construction of Y_i are quite direct. Take a positive cone P_1 of $\pi_1(M)$ containing $\mu_1^{-1}\lambda_1$ and let $Y_1 = P_1 \cap \mathrm{NC}_{\pi_1(M)}(\mu_1^{-1}\lambda_1)$. (We use $\mathrm{NC}_G(g)$ for the normal closure of g in G.) Similarly, $Y_2 = P_2 \cap \mathrm{NC}_{\pi_2(M)}(\mu_2\lambda_2)$, where P_2 is a positive cone of $\pi_2(M)$ containing $\mu_2\lambda_2$.

(d) We remark that the proof of the matching condition involving Y_i uses Dehn surgery and the theorem that the fundamental group of a closed orientable aspherical and irreducible 3-manifold is torsion free.