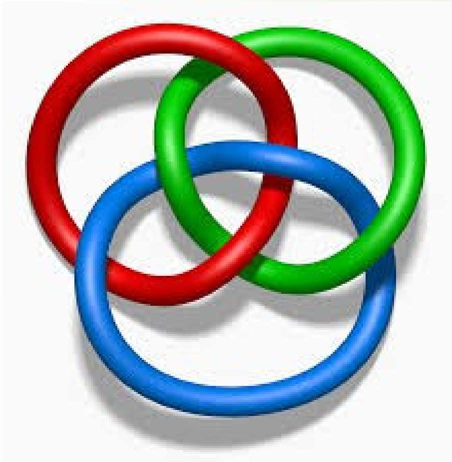
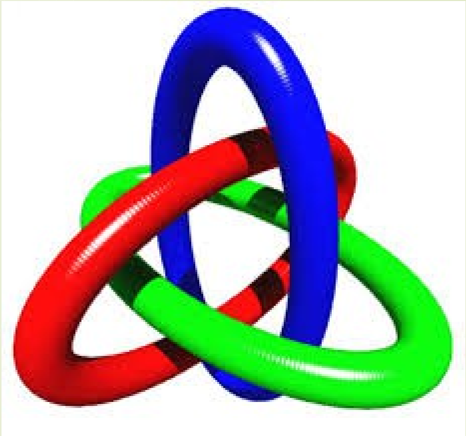
On bi-orderability of Knots February 19th, U. of Manitoba

Joint work with Cody Martin

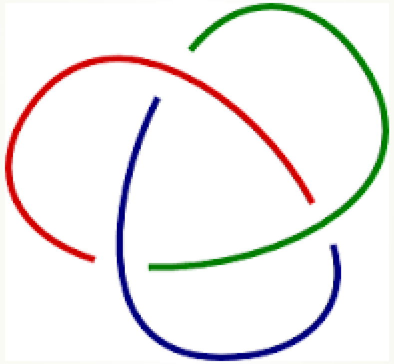
Links





Borromean Rings. It is a Brunnian link.

Knots



Trefoil Knot
Unknotting Number = 1
Genus = 1
Crossing Number = 3
Braid Number = 2
Braid Dimension = 4

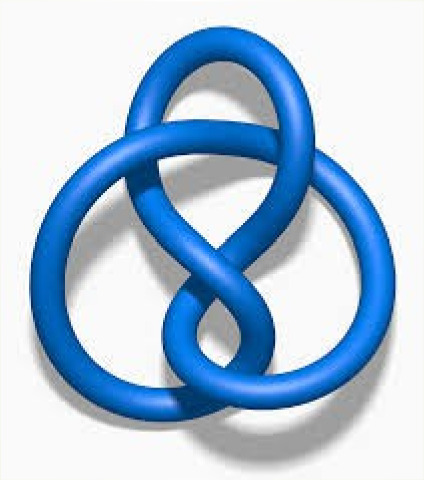
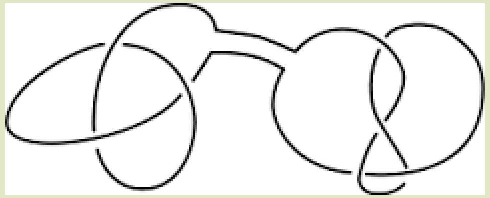


Figure 8 Knot
Unknotting Number = 1
Genus = 1
Crossing Number = 4
Braid Number = 3
Braid Dimension = 4

Connected Sum, Crossing Number



Connected sum of Trefoil and Figure 8.

A knot is called **prime** if it is not a connected sum of two non-trivial knots.

- Crossing number c(K) of a knot K is the smallest possible number of crossings in all possible knot diagrams of K.

- **A Notorious Open Problem: Is it true that c(K#L) = c(K) + c(L)???**

Crossing Number, Prime Knots

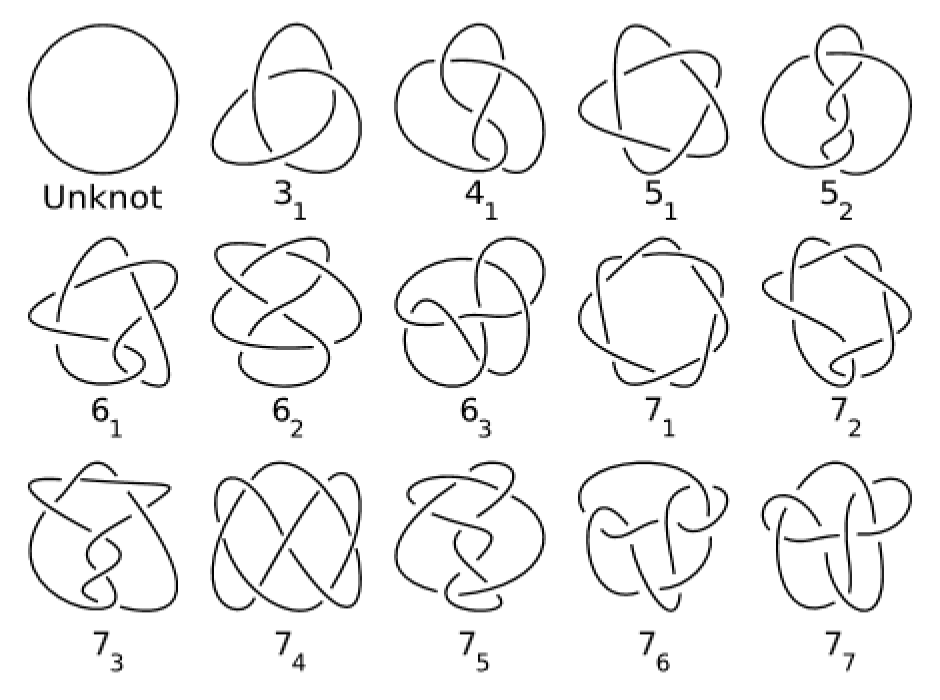
- **A Notorious Open Problem: Is it true that c(K#L) = c(K) + c(L)???**
- Theorem (M.Lackenby, 2009) (1/281)(c(K)+c(L)) < c(K#L) \leq c(K) + c(L)

Table of Prime Knots.

There is no knot with crossing number 1 or 2.

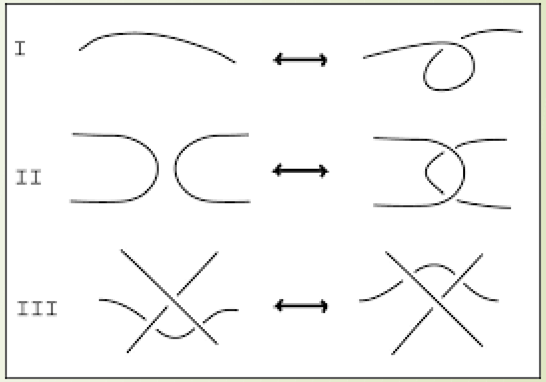
Does 7_4 remind you anything? What???

Open Problem: Is there a prime nugatory knot??



Reidemeister Moves

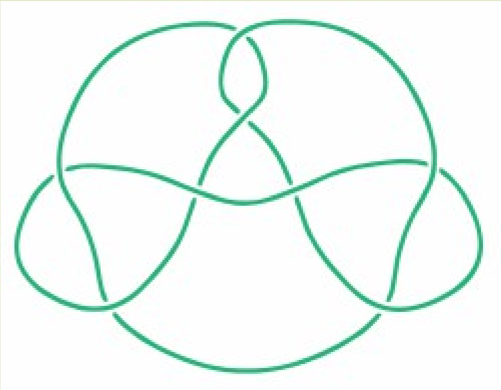
Reidemeister (1926), Alexander & Briggs (1927) Two diagrams of the same Knot can be related by a sequence of Reidemeister knots



Torus Knots, Alternating Knots

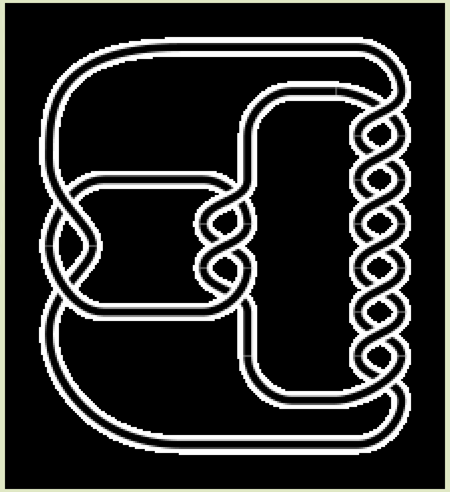
A torus link is determined by two integers p, q.
It is obtained by a circle on an unknotted torus by “wrapping” p times around the parallel and q times around the meridian. If p, q are coprime then one obtains **a torus knot**. A torus knot is trivial iff p=1 or q=1.
Tefoil is a torus knot, Figure 8 is not.

A knot is called **alternating** if it has an alternating knot diagram. All knots with crossing number less than 8 are alternating.
8_{19}, 8_{20}, 8_{21} are not alternating.

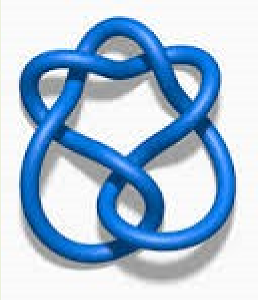


Pretzel Links

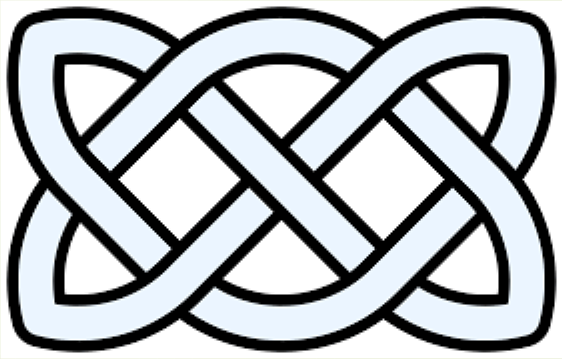
(-2, 3, 7) Pretzel Knot



(5,-1,-1) Pretzel Knot



(p_1,…,p_n) Pretzel Link is a knot iff
both n and all the p_i are odd
or exactly one of the p_i is even.



Tait Conjectures

- The Tait conjectures (1880s) are:

1. Any reduced diagram of an alternating link has the fewest possible crossings.

2. Any two reduced diagrams of the same alternating knot have the same writhe.

3. Given any two reduced alternating diagrams D_1 and D_2 of an oriented, prime alternating link: D_1 may be transformed to D_2 by means of a sequence of certain simple moves called *flypes*.

1&2 proved in 1987 by Thisetwaite, Kaufmann, Murasigi.

3 proved in 1991 by Thisletwaite, Menasco.

Dehn Filling, 3-Manifolds

- **Theorem (Wallace1960, Lickorish 1962)** Every connected, closed, orientable 3-mainifold is a Dehn Filling of a link in a three-sphere.

Using this one can prove

Rokhlin's Theorem: Every 3-manifold as above is a boundary of a 4 manifold.

Fibered Knots

- A knot K is fibered if S^3\K can be fibered over S^1 such each fiber is a Seifert surface of K.

The unknot, the trefoil and the Figure-8 knot are fibered.

Fact 1: (***Rapaport, Ann. of Math, 1960***) **The Alexander polynomial of a fibered knot is monic. Moreover, the degree of the polynomial is half the genus of the knot.**

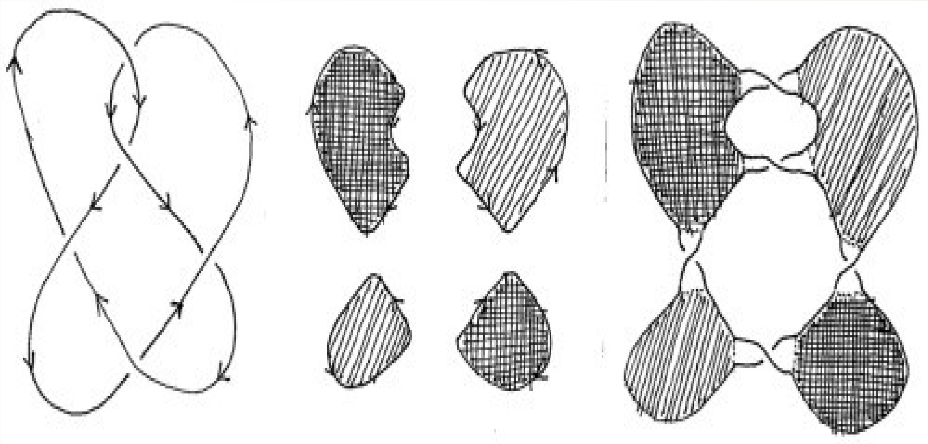
The knot 6_1 is not fibered; its Alexander polynomial equals **-2t^2+5t-2**

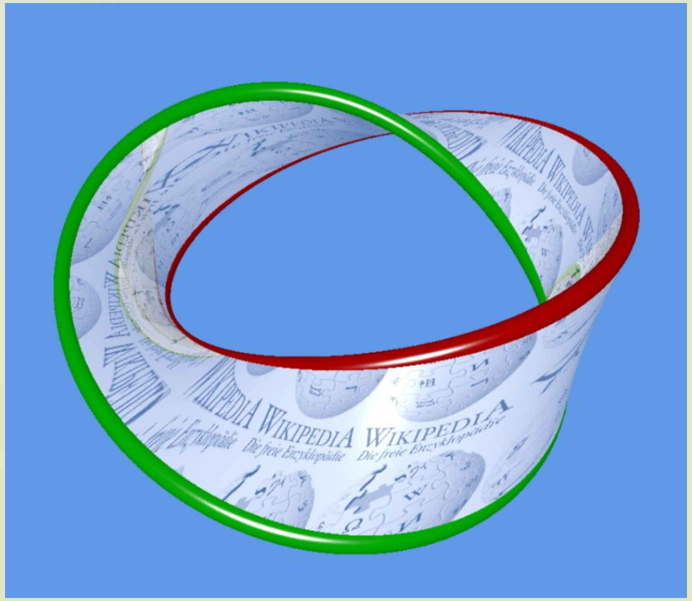
Fact 2: (***Murasigi. Amer.J.Math,1963***) **If the Alexander polynomial of an alternating knot is monic then it is fibered.**

Wirtinger Presentation

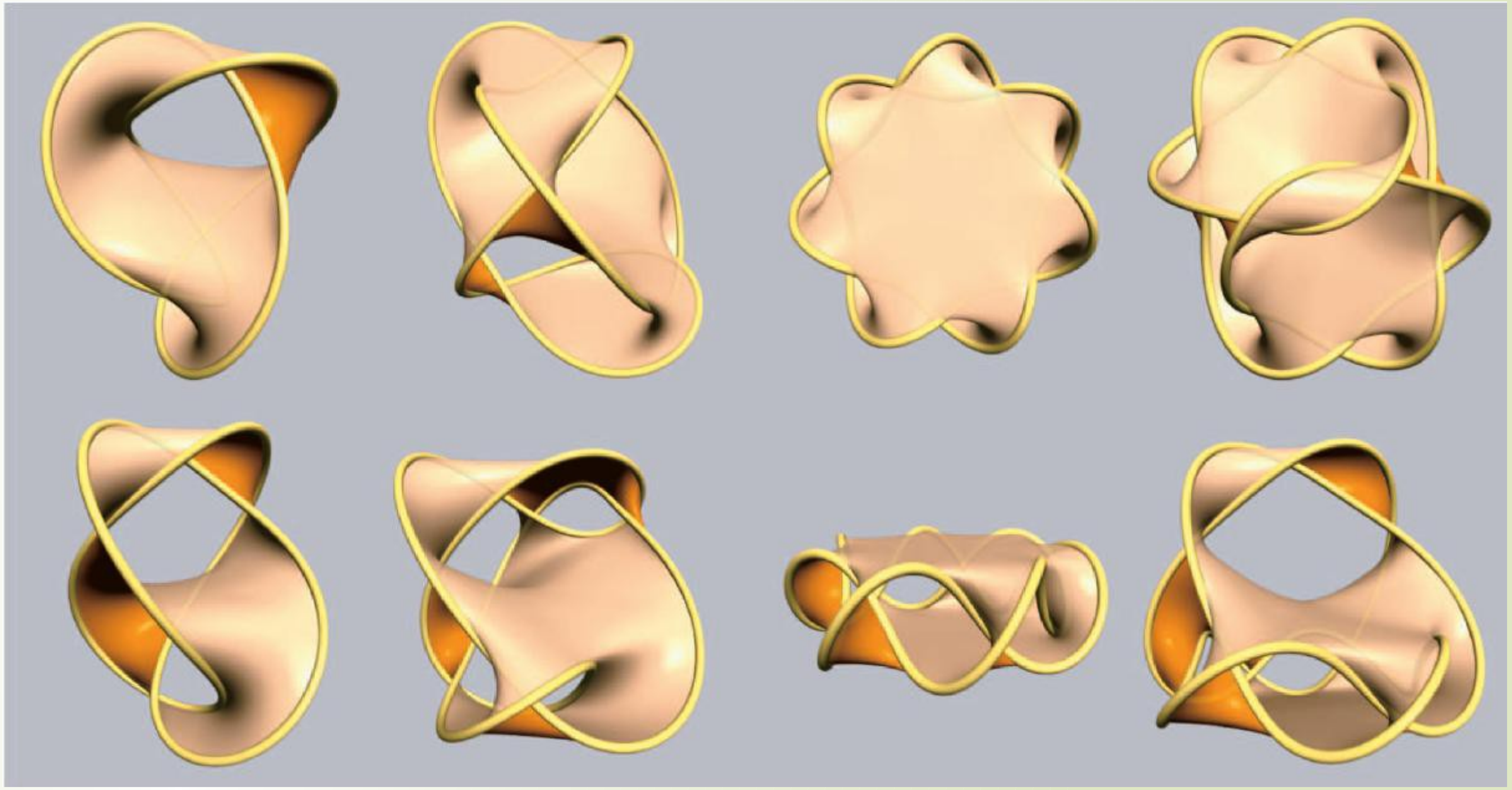
Seifert Surfaces

- Pontriagin 1930, Seifert 1934. Every knot K admits an orientable surface S which bounds K.





Some Seifert Surfaces



Known Results:

- **Howie-Short (1985)** If M is a compact, closed, orientable 3-manifold and H_1(M) is infinite then \pi _1(M) is left-orderable
- **Howie-Short (1985)** Every knot group is left-orderable.
- **Perron-Rolfsen (2003)** Let K be a fibered knot. If the roots of the Alexander polynomial are all real and positive then K is bi-orderable.
- **Clay-Rolfsen (2012)** Let a knot K be bi-orderable and fibered. Then the Alexander polynomial has a positive real root.

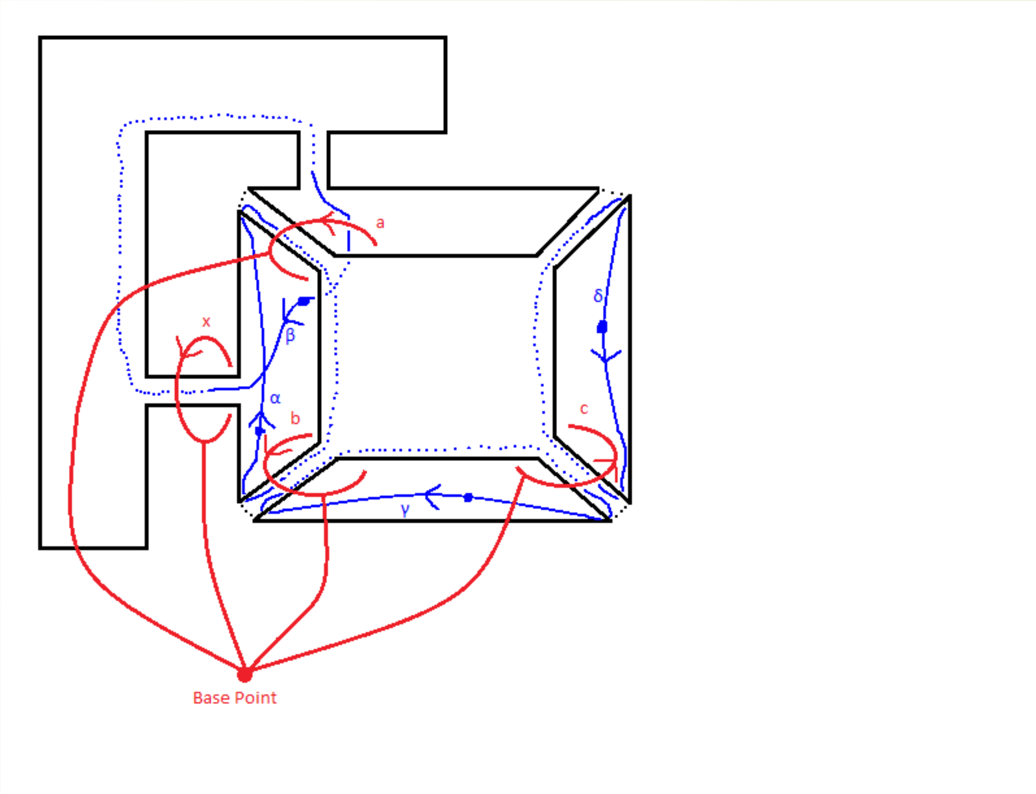
Bi-Orderable: 0_1(The unknot), 4_1(Figure 8), and 6_1.

Non-bi-orderable: 3_1 (Trefoil), 5_1, 5_2, 6_3, 7_1, 7_2, 7_3, 7_4, 7_5, 7_7

The results of **Chiswell-Glass-Wilson (2015)** allow to find more bi-orderable and non-bi-orderable knots with 12 or less crossings.

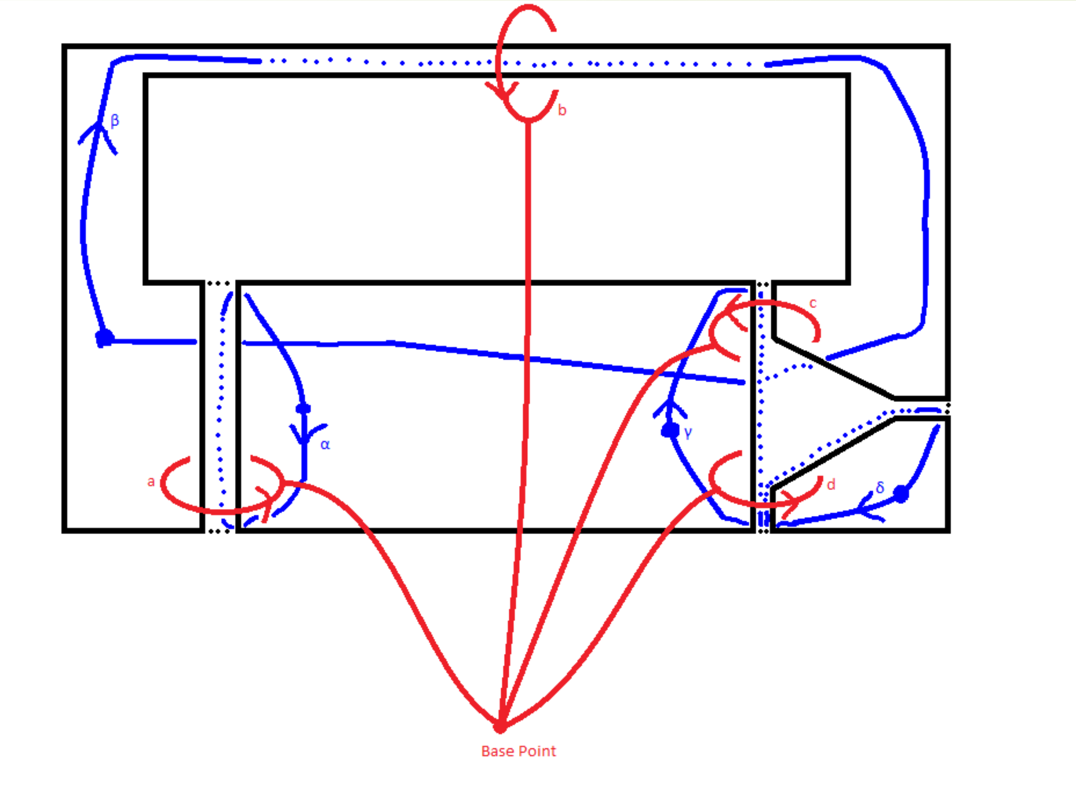
Seifert Surface of 6_2

Computation of the Knot Group



Seifert Surface of 7_6

Computation of the Knot Group



Alexander Polynomial

$\pm t^n$

- Alexander's Definition

Take an oriented diagram of the knot with *n* crossings;
there are $n + 2$ regions of the knot diagram.
Create an incidence matrix of size $(n, n + 2)$. The *n* rows correspond to the *n* crossings, and the $n + 2$ columns to the regions.
The values for the matrix entries are either $0, 1, -1, t, -t$.

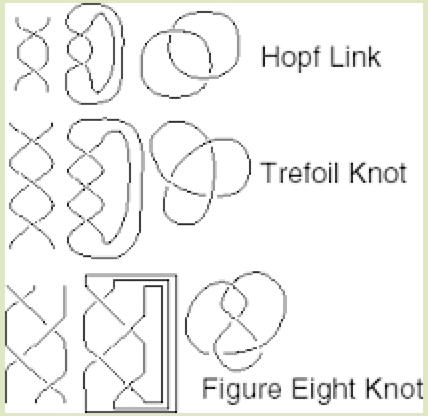
Consider the entry corresponding to a particular region and crossing. If the region is not adjacent to the crossing, the entry is 0. If the region is adjacent to the crossing, the entry depends on its location.
The following table gives the entry, determined by the location of the region at the crossing from the perspective of the incoming undercrossing line.

on the left before undercrossing: $-t$
on the right before undercrossing: 1
on the left after undercrossing: t
on the right after undercrossing: -1

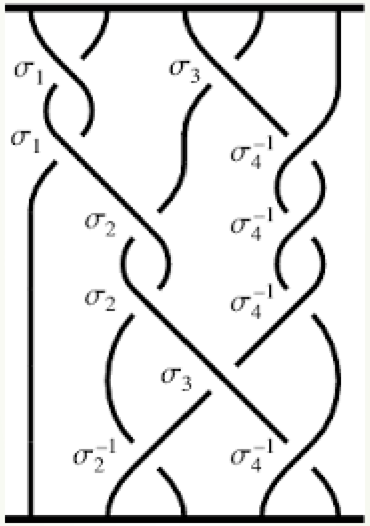
Remove two columns for adjacent regions, and compute the determinant

Alexander's Theorem

- **Every link (in particular, every knot) is a braid closure (i.e. can be obtained from a braid by tying the ends).**
- **Thus for a link L, one can define:**
- ***the braid number* of L is the minimal number of strings needed t form a braid whose closure is L.**
- ***the braid dimension* is the minimal length of a braid whose closure is L.**



Braids



Isotopy classes of braids on n strings form a group B_n – **the braid group on n strings**.

B_1 is a trivial group, B_2 is isomorphic to Z.

The sigmas are **elementary braids. The elementary braids generate B_n**