# 3-Manifold Groups ABC

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# PRELUDE

> Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
>
> — Michael Atiyah

# PRELUDE

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# Prelude

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In dimension 3, algebra almost “determines” geometry and topology.

**Poincaré Conjecture (Perelman, Thurston, etc.)**

Every 3-dimensional manifold which is closed, connected, and has trivial fundamental group, is homeomorphic to $S^3$.

# Deficiency

**Definition**

Let $G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$ be a finitely presented group. The deficiency of the presentation is defined to be $n - k$. The deficiency $\mathrm{def}(G)$ of $G$ is the maximal deficiency of all possible presentations.

# DEFICIENCY

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Some classical results from group (co)homology:

Given a finite presentation $1 \to R \to F_n \to G \to 1$ for $G$,

- $H_1(G) = G/[G,G] = F_n/[F_n,F_n]R$,
- $H_2(G) = (R \cap [F_n,F_n])/[R,F_n]$,
- $\operatorname{def}(G) \leqslant b_1(G) - b_2(G)$.

# DEFICIENCY

$$\begin{aligned}\mathrm{def}(G) &= \max\{\#\text{ generators } -\#\text{ relators in a finite presentation}\}\\ &\leqslant b_1(G) - b_2(G)\end{aligned}$$

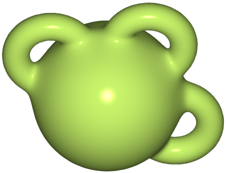
## Examples

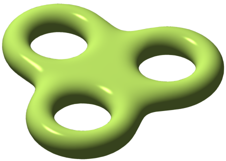
- $\mathrm{def}(F_n) = n$.
- $\mathrm{def}(\mathbb{Z}^2) = 1, \mathrm{def}(\mathbb{Z}^3) = 0$ and $\mathrm{def}(\mathbb{Z}^n) < 0$ for $n \geqslant 4$.
- $|G| < \infty \implies \mathrm{def}(G) \leqslant 0$, for otherwise the abelianization is infinite.
- $\mathrm{def}(\pi_1(\Sigma_g)) = 2g - 1$ with the canonical presentation.
- The only finitely generated abelian groups with $\mathrm{def}(G) = 0$ are $\mathbb{Z}/n\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}^3$.

# Heegaard Splitting

Handlebody $H_g = B^3$ with $g$ copies $I \times D^2$ attached along $(\partial I) \times D^2 \to \partial B_3$

$\partial H_g = \Sigma_g$ surface of genus $g$





# Heegaard Splitting

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A Heegaard splitting of a closed 3-manifold $M$ is a decomposition

$$M = H_1 \cup H_2$$

such that

- $H_1, H_2$ are handlebodies;
- $\partial H_1 = \partial H_2 = H_1 \cap H_2$.

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## Example

- $S^3$ admits a Heegaard splitting with $H_1 = H_2 = B^3$ trivially.
- $S^3$ admits a Heegaard splitting with $H_1 = H_2 = S^1 \times D^2$.

# Heegaard Splitting

**Theorem (Moise, etc.)**

*A topological 3-manifold admits precisely one smooth structure (up to diffeomorphism) and precisely one piecewise structure (up to piecewise-linear homeomorphism).*

In other words, every 3-manifold can be triangulated.

# HEEGAARD SPLITTING

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**Theorem**

*Every closed 3-manifold admits a Heegaard splitting.*

**Sketch of proof.**

Triangulate $M$. $H_1$ is the closure of the regular neighbourhood of 1-skeleton and $H_2$ is the closure of the complement. □

# Heegaard Splitting

Apply the van Kampen theorem to $M = H_1 \cup H_2$:

$$\begin{aligned}\pi_1(M) &= \pi_1(H_1) *_{\pi_1(\Sigma_g)} \pi_1(H_2) \\ &= \langle x_1, \ldots x_g, y_1, \ldots, y_g \mid f_*(a_i) = g_*(a_i), i = 1, \ldots, 2g \rangle\end{aligned}$$

- $\pi_1(H_1) = \langle x_1, \ldots, x_g \rangle$,
- $\pi_1(H_2) = \langle y_1, \ldots, y_g \rangle$,
- $\pi_1(\Sigma_g) = \langle a_1, \ldots, a_{2g} \mid \prod_{i=1}^{g} [a_{2i-1}, a_{2i}] \rangle$,
- $f_*$ is induced by $f : \Sigma_g \to \partial H_1$,
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- $g_*$ is induced by $g : \Sigma_g \to \partial H_2$.

**Corollary**

*Let $M$ be a closed 3-manifold. Then* $\mathrm{def}(\pi_1(M)) \geqslant 0$.

# Prime and Irreducible 3-manifolds

Connected sum

$$M_1 \# M_2 := (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$$

- $B_i = $ 3-balls
- $f = $ an orientation-reversing homeomorphism between $\partial B_1$ and $\partial B_2$

**Definition**

A 3-manifold $M$ is called prime if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.

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**Definition**

A 3-manifold $M$ is called irreducible if every embedded 2-sphere in $M$ bounds a 3-ball in $M$.

$$\begin{aligned} \text{irreducible} &\implies \text{prime} \\ \text{prime} &\implies \text{irreducible or } S^1 \times S^2 \end{aligned}$$

# Prime and Irreducible 3-manifolds

Example of irreducible 3-manifolds:

- $S^3$ (Alexander Theorem)
- Lens spaces $L(p, q)$
- Knot complements $\overline{S^3 - n(K)}$ and most of their Dehn fillings
- Surface bundles over $S^1$ not $S^1 \times S^2$, e.g. mapping torus
- Seifert manifolds, except $S^1 \times S^2$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$

**Theorem (Epstein)**

*A 3-manifold $M$ is Seifert fibered if and only if it is foliated by $S^1$.*

One may interpret "foliated by $S^1$" as "a disjoint union of circles".

# Prime Decomposition Theorem

## Prime Decomposition Theorem (Knerser, Milnor)

Every 3-manifold $M$ with no spherical boundary components can be "uniquely" written as

$$M = M_1 \# \ldots \# M_n,$$

where $M_i$'s are prime. In particular,

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n).$$

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## Corollary

*Let $M$ be closed. If $\pi_1(M)$ is finite, then $M$ is irreducible.*

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## Kneser's Conjecture (Stallings)

If $\pi_1(M) = G_1 * G_2$, then there is a connected sum decomposition $M = M_1 \# M_2$ with $\pi_1(M_i) = G_i$. In particular, $M$ is not irreducible.

# Rigidity

In dimension 3, algebra almost "determines" geometry and topology.

**Theorem**

*Let $M_1, M_2$ be closed 3-manifolds, not lens spaces. If $M_1$ is prime and $\pi_1(M_1) = \pi_1(M_2)$, then $M_1 = M_2$.*

One may compare it with the Mostow rigidity theorem.

**Theorem (Mostow, Prasad, Marden)**

*If two hyperbolic 3-manifolds with finite volume have isomorphic fundamental groups, they are isometric.*

This theorem implies in particular that the geometry of finite-volume hyperbolic 3-manifolds is determined by their topology.

# Sphere Theorem

Here is a powerful tool.

**Sphere Theorem (Papakyriakopoulos)**

Let $M$ be a 3-manifold. If $\pi_2(M)$ is not trivial, then there is an embedded 2-sphere $S^2 \hookrightarrow M$ representing a non-trivial element in $\pi_2(M)$. In particular, $M$ is reducible.

With some work, we see the sphere theorem implies the following.

**Theorem**

*Let $M$ be a 3-manifold and $\widetilde{M} \to M$ be a covering. Then $M$ is irreducible if and only if $\widetilde{M}$ is irreducible.*

# Sphere Theorem

## Corollary

*If $M$ is an irreducible 3-manifold, then*

- $\pi_2(M)$ *is trivial;*
- $\pi_1(M)$ *is finite if and only if* $\pi_3(M)$ *is non-trivial; in particular,* $M = B^3$ *or* $M$ *is closed;*
- $\pi_1(M)$ *is infinite if and only if* $\pi_3(M)$ *is trivial; in this case,* $\pi_1(M)$ *is torsion-free and* $M$ *is aspherical, and if* $M$ *is closed,* $M$ *is* $K(\pi, 1)$.

## Sketch of proof.

The universal cover $\widetilde{M}$ has the same higher homotopy groups $\pi_n$ as $M$ for $n \geqslant 2$. Also note that $\pi_1(M)$ is infinite if and only if $\widetilde{M}$ is non-compact. The conclusions follow from the Hurewicz theorem applied to $\widetilde{M}$. □

# DEFICIENCY OF 3-MANIFOLD GROUPS

**Corollary**

*If $M$ is closed and irreducible, then* $\operatorname{def}(\pi_1(M)) = 0$.

**Proof.**

It remains to see $\operatorname{def}(\pi_1(M)) \leqslant 0$. It is clearly true if $\pi_1(M)$ is finite. If $\pi_1(M)$ is infinite, then the previous corollary tells us that $M$ is the classifying space of $\pi_1(M)$. Hence, $H_k(\pi_1(M)) = H_k(M)$ and so $b_2(\pi_1(M)) = b_2(M) = b_1(M) = b_1(\pi_1(M))$ by duality theorems. □

# Deficiency of 3-manifold Groups

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**Corollary**

*The possible abelian fundamental groups of a closed 3-manifold are* $\pi_1(S^3) = 1, \pi_1(S^1 \times S^2) = \mathbb{Z}, \pi_1(T^3) = \mathbb{Z}^3$ *and* $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$.

# Incompressible Surface

Recall that to study an arbitrary 3-manifold, we cut it into irreducible pieces along spheres. It turns out irreducible 3-manifolds can be further decomposed.

**Definition**

Let $S \subset M$ be a properly embedded surface, i.e. $\partial S \subset \partial M$. We say $S$ is incompressible if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective and $S$ does not bound a 3-ball.

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**Loop Theorem (Papakyriakopoulos)**

Let $M$ be a 3-manifold and $F \subset \partial M$ is a boundary component. If the induced homomorphism $\pi_1(F) \to \pi_1(M)$ is not injective, then there is a proper embedding $g : (D^2, \partial D^2) \to (M, \partial M)$ such that $g(\partial D^2)$ represents a non-trivial element in $\ker(\pi_1(F) \to \pi_1(M))$.

# Loop Theorem

**Corollary**

*Let $M$ be a 3-manifold. There exist compact 3-manifolds $N_1, \ldots, N_m$ whose boundary components are incompressible and a free group $F$ such that $\pi_1(M) = \pi_1(N_1) * \cdots * \pi_1(N_m) * F$.*

**Sketch of proof.**

Cut $N$ along the disks obtained by the Loop Theorem. □

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**Sketch of proof.**

Cut $N$ along the disks obtained by the Loop Theorem. □

**Corollary**

*The fundamental group $\pi_1(M)$ is infinite cyclic if and only if $M = S^1 \times S^2$ or $M = S^1 \times D^2$.*

# JSJ Decomposition

## JSJ Decomposition Theorem (Jaco-Shalen, Johannson)

Let $M$ be an irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ such that each component of $M$ cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.

Recall that a 3-manifold is Seifert fibered if it is foliated by circles. And a 3-manifold $N$ is atoroidal if any map $T \to N$ from a torus $T$ to $N$ which induces a monomorphism $\pi_1(T) \to \pi_1(N)$ can be homotoped into the boundary of $N$. We can say something more about the atoroidal piece.

# Geometrization Theorem

We say a 3-manifold is <u>spherical</u>, resp. <u>hyperbolic</u>, if it admits a complete metric of constant curvature $+1$, resp. $-1$.

### Elliptization Theorem

Every closed 3-manifold with finite fundamental group is spherical.

Recall that a closed 3-manifold $M$ with finite fundamental group has universal cover $S^3$. Hence $\pi_1(M)$ is a finite subgroup of $\mathrm{SO}(4)$.

### Hyperbolization Theorem

Let $N$ be an irreducible 3-manifold with empty or toroidal boundary. Suppose that $N$ is atoroidal and not homeomorphic to $S^1 \times D^2, T^2 \times I$, or $K \tilde{\times} I$. If $\pi_1(N)$ is infinite, then $N$ is hyperbolic.

# Geometrization Theorem

## Geometrization Theorem

Let $M$ be an irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ in $M$ such that each component of $M$ cut along $T_1 \cup \cdots \cup T_m$ is hyperbolic or Seifert fibered. Furthermore, any such collection of tori with a minimal number of components is unique up to isotopy.

## Proof.

This is a direct consequence of the JSJ-Decomposition Theorem with the Elliptization Theorem and the Hyperbolization Theorem and the aforementioned facts that spherical 3-manifolds as well as $S^1 \times D^2, T^2 \times I$, and $K \tilde{\times} I$ are Seifert fibered, and that hyperbolic 3-manifolds are atoroidal. □

This theorem can be further developed into the Thurston's geometrization conjecture (now a theorem).

# Geometrization Theorem

There are many consequences (and/or side products) of the Thurston's geometrization conjecture. Here is one of them.

**Theorem**

*Let $M$ be an irreducible 3-manifold with empty or toroidal boundary. Suppose $\pi_1(M) = A \times B$ where $A$ is infinite and $B$ is non-trivial. Then $M = S^1 \times \Sigma$ where $\Sigma$ is a surface.*

Note that a free product of non-trivial groups is never a direct product.

# Coherence

A group is called coherent if each of its finitely generated subgroups is finitely presented.

For example, $F_2 \times F_2$ is not coherent. For $n \neq 3$, $\mathrm{SL}(n, \mathbb{Z})$ is known to be coherent; for $n = 3$, it is currently unknown.

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**Compact Core Theorem (Scott)**

If $Y$ is a 3-manifold such that $\pi_1(Y)$ is finitely generated, then $Y$ has a compact submanifold $M$ such that the induced map $\pi_1(M) \to \pi_1(Y)$ is an isomorphism.

**Corollary**

*Every 3-manifold group is coherent.*

# LINEARITY

Let $R$ be a commutative ring with unity. We say that $G$ is linear over $R$ if there exists an embedding $G \to \mathrm{GL}(n, R)$ for some $n$.

An irreducible 3-manifold $M$ with empty or toroidal boundary

- is Seifert fibered;
  - fundamental groups of Seifert fibered manifolds are linear over $\mathbb{Z}$.
- is hyperbolic;
  - $\pi_1(M)$ admits a faithful representation $\pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$, which lifts to a faithful representation $\pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$.
- admits an incompressible torus.

**Conjecture (Thurston)**

All 3-manifold groups are linear.

# Residually Finiteness

A group $G$ is called residually finite if for every $g \in G \setminus \{1\}$, there is a finite group $H$ and a homomorphism $f : G \to H$ such that $f(g)$ is non-trivial.

The Baumslag-Solitar group $BS(2,3)$ is not residually finite. The infinite dihedral group is not residually finite either.

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**Theorem (Mal'cev-Selberg)**

*Finitely generated linear groups are residually finite.*

**Theorem (Hempel)**

*All 3-manifold group are residually finite.*