Math 3390

Introduction to topology, Assignment 4.

Due November 23.

Question 1: Suppose that $\{A_i\}$ are connected subspaces of a space X, satisfying $A_i \cap A_0 \neq \emptyset$ for all $i \in I$. Show that $\bigcup_{i\in I} A_i$ is connected.

Question 2: Prove the lemma we used in the proof of connectedness of product spaces. Namely, suppose that $\{X_i\}_{i\in I}$ are connected spaces and fix $(x_i) \in \prod_{i\in I} X_i$. For every finite subset $T \subset I$, set $C(T) = \prod_{i\in I} A_i$, where $A_i = \{x_i\}$ if $i \notin T$, and $A_i = X_i$ if $i \in T$. Prove that

$$Y = \bigcup_{T \subset I finite} C(T)$$

is a dense subset of $\prod_{i\in I} X_i$.

Question 3: Show that if D_i is dense in X_i for all $i \in I$, then $\prod_{i\in I} D_i$ is dense in $\prod_{i\in I} X_i$ (with the product topology). Is the same true for $\prod_{i\in I}^{Box} X_i$?

Question 4: If X is connected, show that the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a connected subset of X (with the subspace topology).

Question 5: Use connectedness to show that for any homeomorphism $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x)) = x$ for all $x \in \mathbb{R}$, there must exist $x_0 \in \mathbb{R}$ with $f(x_0) = x_0$.

Question 6: Suppose that $U \subset \mathbb{R}^n$ has nonempty interior, where $n \geq 2$. Show that there is no embedding $f : U \to S^1$. (Hint: Restrict your attention to a basic open subset of U in order to make things easier. Remember that path connected implies connected, so if you need a particular set to be connected it might be easier to argue that it is path connected.)

Question 7: Let X be a countable set with the cofinite topology. Show that X is locally connected but not locally path connected.