Math 3390
Introduction to topology, Assignment 3.

Due November 9.

Question 1: A property P of a space X is called *weakly hereditary* if all of the closed subspaces of X, with respect to the subspace topology, have property P. Show that the following property is weakly hereditary: For every pair of close disjoint subsets E, F of X, there exist open disjoint subsets U, V of X with $U \subset E$ and $V \subset F$ (this property is called *normality*).

Question 2: In the gluing lemma that we proved in class, we required two subsets A and B to both be closed.

1. Show that the gluing lemma still holds if we replace "closed" with "open".
2. Show that if we omit the requirement that the sets A and B be open or closed, then the union of the maps f and g need not be continuous even if f and g are continuous (so the gluing lemma fails).

Question 3: Let $A = X = [0, 1]$ and let $Y = [2, 3]$. Define a map $f : A \to Y$ by $f(x) = 2$ if $x < 1$, and $f(1) = 3$. Show that the resulting space $X \cup_f Y$ cannot be embedded in $\mathbb{R}^n$ for any n. (Hint: Analyze the topology on $X \cup_f Y$. Is it missing any properties that every subspace of $\mathbb{R}^n$ must have?).

Question 4: Let

$$S^n = \left\{ (x_0, \ldots, x_n) : \sum_{i=0}^{n} x_i^2 = 1 \right\} \subset \mathbb{R}^{n+1}$$

equipped with the subspace topology. Prove that S^n is an n-manifold, by following the example of $\mathbb{R}^3$ from class and carefully checking any details that were omitted. You may use, without proof, the fact: A function $f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is continuous if and only if its component functions $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ are continuous.

Question 5: Define an equivalence relation on S^n by declaring every point to be equivalent to its antipodal point, so the equivalence classes are $[\mathbf{x}] = \{\mathbf{x}, -\mathbf{x}\}$. Use the fact that S^n is an n-manifold to show that $S^n/\sim$ is an n-manifold as well. The manifold $S^n/\sim$ is commonly called $\mathbb{R}\mathrm{P}^n$. Note: For condition (iii) of a manifold, use the maps from the previous question to find a neighbourhood of $[\mathbf{x}] \in \mathbb{R}\mathrm{P}^n$ which is homeomorphic to an open subset of $\mathbb{R}^n$.

Question 6:

1. Show that if $X_1, \ldots, X_n$ are separable, then so is $\prod_{i=1}^{n} X_i$.
2. Show that if $X_1, \ldots, X_n$ are second countable, then so is $\prod_{i=1}^{n} X_i$.

Question 7: Suppose that I is infinite and U is a proper open subset of X_i for all i. Show that $\prod_{i \in I} U_i$ is not an open subset of $\prod_{i \in I} X_i$.

Challenge problem (not for credit): Prove that a normality is not hereditary.

Challenge problem (not for credit): Prove that $\prod_{i=1}^{\infty} \mathbb{R}_i$ is not metrizable.

Not a math problem, but some general mathematical education: When we define manifolds, we require that every point has a neighbourhood U that is homeomorphic to $\mathbb{R}^n$, call the homeomorphism ϕ_U. This means that when two neighbourhoods overlap, say $U \cap V$, there is a map $\mathbb{R}^n \to U \cap V \to \mathbb{R}^n$, given by composing ϕ_V^{-1} with $\phi(U)$. Because this is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, we could ask for it to have some particularly nice properties aside from being continuous. For example, we could insist that all such "overlap maps" are differentiable, or smooth (infinitely differentiable), or piecewise-linear.

Then a fundamental question becomes: When I ask for these maps to be smooth, say, instead of simply continuous, does that make anything different? I.e. are there some manifolds for which the maps $\mathbb{R}^n \to U \cap V \to \mathbb{R}^n$ cannot all be smooth, only continous is possible?

The surprise is that the answer depends on the dimension of the manifold. In dimensions $n = 1, 2, 3$ all these definitions are equivalent. In dimensions 4 and higher, you start running into differences: Some manifolds have all maps $\mathbb{R}^n \to U \cap V \to \mathbb{R}^n$ continuous, but it is impossible to modify them in order to make them smooth, and figuring out which dimensions behave differently than dimensions $1, 2$ and 3 was at the core of a long-open problem that was solved in 2013:

`https://www.quantamagazine.org/20150113-a-proof-that-some-spaces-cant-be-cut/`