Math 3390

Introduction to topology, Assignment 2.

Due October 26.

Question 1: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that the $\epsilon - \delta$ definition of continuity of f is equivalent to our definition of continuity in terms of local bases.

Question 2: Fix a set X. Let X_0 denote the set X equipped with a topology τ_0, and let X_1 denote the set X equipped with a topology τ_1. Consider the identity map $id : X_0 \to X_1$.

1. Show that id is continuous if and only if $\tau_1 \subset \tau_0$.
2. Show that id is a homeomorphism if and only if $\tau_0 = \tau_1$.

Question 3: By the Baire category theorem, $\mathbb{R}$ cannot be written as a countable union of closed sets having empty interior. Show that "closed" is necessary in this statement, by expressing $\mathbb{R}$ as a countable union of sets (not necessarily closed) having empty interior.

Question 4: Given a family $\{X_i\}_{i \in I}$ of topological spaces, prove that the projection maps

$$p_i : \prod_{i \in I} X_i \to X_i$$

are open maps.

Question 5: Consider the space $X = [0,1]/\sim$, where $\sim$ is the relation whose equivalence classes are all singletons, except for the equivalence class $\{0,1\}$. (So 0 is identified with 1, and there are no other identifications). Show that X is homeomorphic to the subspace $\{(x,y) \mid x^2 + y^2 = 1\}$ of $\mathbb{R}^2$.

Question 6: Quotient maps need not be open maps or closed maps. Here is an example that shows this fact: Let $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the projection onto the first coordinate. Let A be the sucspace of $\mathbb{R} \times \mathbb{R}$ consisting of (x,y) with either $x \geq 0$ or $y = 0$, or both. Let $q : A \to \mathbb{R}$ denote the restriction of p_1 to the set A. Show that q is a quotient map which is neither open nor closed.

Question 7: Set $X = \{0,1\}$. List all the topologies on the set X, and show that for every one of them arises as a quotient of $\mathbb{Q}$ (here, $\mathbb{Q} \subset \mathbb{R}$ is given the subspace topology).

Challenge problem (not for credit): Show that if D is a countable dense subset of $\mathbb{R}$, then there is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at each point $d \in D$ and discontinuous at each point $x \in \mathbb{R} \setminus D$. (Sketch: The proof proceeds in two steps. First show that the set C of points where f is continuous can be expressed as a countable intersection of open sets in $\mathbb{R}$. This can be done by setting U_n to be the union of all sets U of $\mathbb{R}$ satisfying $diameter(f(U)) < 1/n$ and showing $C = \bigcap U_n$. Then show that D cannot be expressed as a countable intersection of open sets in $\mathbb{R}$. This can be done by setting $D = \bigcap W_n$ where W_n is open, setting $V_d = \mathbb{R} \setminus \{d\}$ and showing that both W_n and V_d are dense in $\mathbb{R}$.)