Math 3390
Topology 1, Assignment 1.

Question 1: (Munkres) Let A, B and A_i denote subsets of a topological space X. Prove the following:

1. If $A \subset B$, then $\overline{A} \subset \overline{B}$.
2. $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.
3. $\bigcup_{i \in I} \overline{A_i} \subset \overline{\bigcup_{i \in I} A_i}$ and give an example showing that the two sets are not always equal.

Question 2: (Munkres) Let A, B and A_i denote subsets of a topological space X. Determine whether or not the following are true, and if equality fails, determine whether or not inclusion (either $\subset$ or $\supset$) holds in general.

1. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
2. $\overline{\bigcap_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$.

Question 3: Which of the following spaces are Hausdorff? In each case, justify your answer.

1. $\mathbb{Z}$, with the toplogy generated by sets of the form $S(a, b) = \{an + b \mid a, b \in \mathbb{Z}, a \neq 0\}$.
2. $\mathbb{Q}$, with the co-countable topology.
3. $\mathbb{R}^2$, with the lexicographic order topology (i.e. declare $(a, b) < (c, d)$ if either $a < c$ or $a = c$ and $b < d$, then give $\mathbb{R}^2$ the resulting order topology).
4. $\mathbb{R}$ with the K-topology.

Question 4: (Moderately difficult) Given a topological space (X, τ), define a topology τ' on $X \times X$ to be generated by all sets of the form $U \times V \subset X \times X$ where $U, V \in \tau$.

Prove that (X, τ) is Hausdorff if and only if the **diagonal**

$$\Delta = \{(x, x) : x \in X\}$$

is a closed subset of $(X \times X, \tau')$.

Question 5: (Moderately difficult) Let $f : X \to Y$ be a surjective, continuous, open map (open means that if U is open, then so is $f(U)$). Show that if X is either first countable or second countable, then Y has the same property.

Question 6: (Difficult) Suppose that $D \subset X$ satisfies $\overline{D} = X$, let Y be a Hausdorff space and let $g, f : X \to Y$ be continuous maps. Show that if $f(d) = g(d)$ for all $d \in D$, then $f(x) = g(x)$ for all $x \in X$. Give an example to show that the conclusion does not always hold if Y is not Hausdorff.

Question 7: (Difficult, Steen & Seebach) Consider the set $\mathbb{N} \times \mathbb{N}$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$). Define a topology τ on $\mathbb{N} \times \mathbb{N}$ by declaring a set U to be in τ if and only if one of the following is true:

- $(0, 0) \notin \tau$.
- If $(0, 0) \in U$, then for all but finitely many $m \in \mathbb{N}$ the set $\{n \mid (m, n) \notin U\}$ is finite. In other words, in an open set U there can only be finitely many "columns" $\{(m, n)\}_{m \in \mathbb{Z}}$ that are missing an infinite number of points.

Prove that this defines a topology. Show that it is not first countable. (Hint: Try making a local basis at $(0, 0)$, feel free to consult Steen & Seebach "Counterexamples in Topology" page 54, but if you use this or any other reference, be sure to cite it. Moreover, convince me that you understand the argument by using your own notation and explanations).