Math 3322
Algebra 3, Assignment 4
Due March 19 at the start of class.

1. Prove that $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$ is the trivial group. (Hint: Since every positive element of $\mathbb{R}$ is a square, it follows that an automorphism of $\mathbb{R}$ sends positives to positives (explain why) and hence that it preserves the order of $\mathbb{R}$ (show this). Now given any real number, show that it can be trapped between suitably close rational numbers. As every element of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$ fixes the rationals and preserves order, we conclude that...)

2. Let K be a field. The following sequence of exercises will allow us to thoroughly understand the Galois group $\mathrm{Aut}_K(K(x))$ by giving an explicit description of its elements. Let $f(x)/g(x) \in K(x) \setminus K$ be given, where $f(x)$ and $g(x)$ are relatively prime in $K[x]$. Consider the extension $K(x)$ of K.

   (a) Prove that x is algebraic over $K(f(x)/g(x))$ and that
   $$[K(x) : K(f(x)/g(x))] = \max\{\deg(f), \deg(g)\}.$$
   (Sketch: Show that the element x is a root of the polynomial $\phi(y) = (f(x)/g(x))g(y) - f(y) \in K(f(x)/g(x))[y]$, and show that this polynomial has degree $\max\{\deg(f), \deg(g)\}$. Then show ϕ is irreducible as follows: We know that $f(x)/g(x)$ is transcendental over K (cite our last assignment). Replacing $f(x)/g(x)$ by the indeterminate z (for convenience) we are reduced to showing that $\phi(y) = zg(y) - f(y) \in K(z)[y]$ is irreducible. But ϕ is irreducible in $K(z)[y]$ provided it is irreducible in $K[z][y]$ (this fact is a generalization of something we already saw, namely that irreducibility in $\mathbb{Z}[x]$ implies irreducibility in $\mathbb{Q}[x]$, which is stated in full generality as Lemma III.6.13 in Hungerford). Explain that irreducibility in $K[z][y]$ follows from ϕ being linear and f, g being relatively prime.)

   (b) If $E \neq K$ is an intermediate field, then $[K(x) : E]$ is finite.

   (c) Show that the assignment $\tau(x) = f(x)/g(x)$ extends to a homomorphism $\tau : K(x) \to K(x)$ such that $\tau(\phi/\psi) = \phi(f(x)/g(x))/\psi(f(x)/g(x))$. Show that τ is an automorphism if and only if $\max\{\deg(f), \deg(g)\} = 1$.

   (d) Show that $\mathrm{Aut}_K(K(x))$ consists of all those automorphisms τ that arise, as in part (c), from an assignment
   $$\tau(x) = (ax + b)/(cx + d)$$
   where $a, b, c, d \in K$ and $ad - bc \neq 0$.

3. We investigate the field extension of K by $K(x)$, and how it depends on the field K.

   (a) Show that if K is an infinite field, then $K(x)$ is Galois over K. (Hint: If $K(x)$ is NOT Galois over K, then $K(x)$ is finite dimensional over the fixed field E of $\mathrm{Aut}_K(K(x))$ by part (b) of the previous question. But part (d) of the previous question shows that $\mathrm{Aut}_E(K(x) = \mathrm{Aut}_K(K(x))$ is infinite, which is a contradiction because...)

   (b) If K is finite, then $K(x)$ is NOT Galois over K. (Hint: If $K(x)$ were Galois over K, then $\mathrm{Aut}_K(K(x))$ would be infinite (why?), but by part (d) above...)

4. Prove that no finite field is algebraically closed. (Hint: Use a list of all the elements in the finite field K to cook up a polynomial in $K[x]$ that has no root in K).