Math 3322
Algebra 3, Assignment 3
Due February 26 at the start of class.

1. Prove that the following statements are equivalent:

   (a) Every finite simple group of odd order is isomorphic to $\mathbb{Z}_p$ for some prime p.

   (b) Every finite group of odd order is solvable.

   It turns out, in fact, that both of these statements are true. This result is called the Feit-Thompson theorem, a "simplified" version of the proof comprises two books.

2. Let $K \subset F$ be fields. Suppose that u is transcendental over K. Show that every element of $K(u)$ which is not in K is transcendental over K.

3. Show that $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$. Next, let θ denote any root of this polynomial. Then $\{1, \theta, \theta^2\}$ is a basis of $\mathbb{Q}(\theta)$ as a vector space over $\mathbb{Q}$. Calculate representations of $(1+\theta)(1+\theta+\theta^2)$ and $\dfrac{1+\theta}{1+\theta+\theta^2}$ relative to this basis.

4. Show that the minimal polynomial for $\sqrt[6]{2}$ over $\mathbb{Q}(\sqrt{2})$ is $x^3 - \sqrt{2}$. (Hint: Try finding relevant fields $K \subset E \subset F$ and using the formula $[F : K] = [F : E][E : K]$.)

5. Prove that the subfields $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ of $\mathbb{C}$ are not isomorphic as fields, but that they are isomorphic as vector spaces over $\mathbb{Q}$.

6. Let E_1 and E_2 be subfields of F and X a subset of F. If every element of E_1 is algebraic over E_2, then every element of $E_1(X)$ is algebraic over $E_2(X)$.

   Hint: Show $E_1(X) \subset (E_2(X))(E_1)$ and use the following theorem:

   Theorem 1. *If F is an extension field of K and X is a subset of F such that $F = K(X)$ and every element of X is algebraic over K, then F is an algebraic extension of K.*