Math 3322
Algebra 3, Assignment 1
Due January 22 at the start of class.

1. Prove each of the claims in the following theorem:

   Theorem 1. *If* $\{G_i \mid i \in I\}$ *is a family of groups, then*

   - $\prod_{i\in I}^{w} G_i$ *is a normal subgroup of* $\prod_{i\in I} G_i$.
   - *For each* $k \in I$ *the map* $i_k : G_k \to \prod_{i\in I}^{w} G_i$ *given by* $i_k(g) = f$, *where* $f(j) = e_j$ *for all* $j \neq k$ *and* $f(k) = g$ *for all* $g \in G_k$ *is a one-to-one homomorphism.*
   - *For each* $k \in I$, *the subgroup* $i_k(G_k)$ *is normal in* $\prod_{i\in I}^{w} G_i$.

2. Find a pair of groups G, H such that $G \times H$ does not satisfy the universal property of the external (weak) direct product. Specifically, show that $G \times H$, together with the obvious homomorphisms $i_G : G \to G \times H$ and $i_H : H \to G \times H$, does not satisfy the following: For every group K with homomorphisms $f_G : G \to K$ and $f_H : K \to H$ there exists a unique homomorphism $f : G \times H \to K$ such that $f \circ i_G = f_G$ and $f \circ i_H = f_H$.

3. Consider a family of groups $\{G_i \mid i \in \mathbb{N}\}$ where each group G_i is isomorphic to a copy of $\mathbb{Z}$. Show that $\bigoplus_{i\in\mathbb{N}} G_i$ is not isomorphic to $\prod_{i\in\mathbb{N}} G_i$ (Hint: It may help to do the next question first, in order to get a feel for $\bigoplus_{i\in\mathbb{N}} G_i$).

4. Let $\mathbb{Q}_{>0}$ denote the positive rational numbers with multiplication as the operation. Show that $\mathbb{Q}_{>0}$ is isomorphic to $\bigoplus_{i\in\mathbb{N}} G_i$ from the previous question (Hint: To define your map, consider writing each element of $\mathbb{Q}_{>0}$ as a product of powers of primes).

5. Suppose that $\{x_1, x_2, x_3\}$ are the generators of a free group F. Let N be the smallest normal subgroup of F containing x_3, meaning that if K is any other normal subgroup of F containing x_3, then $N \subset K$. Show that F/N is free. (Hint: Verify that F/N satisfies the required universal property, with x_1N, x_2N as its generating set. You will need to use the fact that $N \subset K$ implies that there is a quotient map $F/N \to F/K$).

6. Show that the group $\langle a, b \mid a^2 = b^3 = a^{-1}b^{-1}ab = e\rangle$ is isomorphic to a cyclic group of order 6.