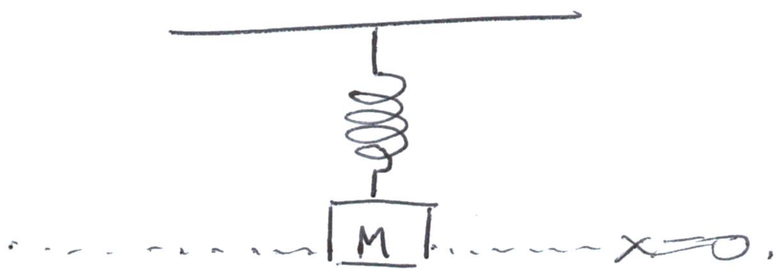
§15.10 Applications.

Note: We will skip operator methods (§15.9). Instead cover applications (15.10) in more detail.

Long example: Mass on a spring.

Suppose we have a mass M suspended on a spring:



Hooke's law says that for some constant k depending on the spring, the force exerted by the spring is

$$F_{spring} = -kx$$

where x is the amount that the spring has been stretched.

Gravity also acts on M with force Mg, $g = 9.81 m/s^2$, and the spring stretches until

$$-F_{spring} = F_{gravity}$$

ie. $$kx = Mg.$$

As our $x=0$ position, we take this equilibrium point.

Let s denote the amount that the spring is stretched when the mass hangs in equilibrium, so

$$ks - Mg = 0.$$

When the mass moves up/down by x, the spring exerts a force of $k(s-x)$.

Then Newton's Law $F = ma$ gives

$$M\frac{d^2x}{dt^2} = k(s-x) - Mg$$

$$= -kx + \underbrace{(ks - Mg)}_{0}$$

So our DE is

$$M\frac{d^2x}{dt^2} + kx = 0.$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{M}x = 0.$$

So the complementary equation is

$$m^2 + \frac{k}{M} = 0$$

or $m = \pm i\sqrt{\frac{k}{M}}$, we know $k > 0$ and $M > 0$ so we get imaginary roots.

Then the behaviour of M is:

$$x(t) = C_1\cos\left(\sqrt{\frac{k}{M}}\,t\right) + C_2\sin\left(\sqrt{\frac{k}{M}}\,t\right).$$

Example: A 2kg mass is suspended on a spring with $k = 16\,N/m$. The mass is raised to a height of 10cm above equilibrium and released, what equation describes its motion?

Solution: Here, $k = 16$, $M = 2$ so $\sqrt{\frac{k}{M}} = \sqrt{\frac{16}{2}} = \sqrt{8}$

$= 2\sqrt{2}$.

So the motion is

$$y(x) = c_1 \cos(2\sqrt{2}\,t) + c_2 \sin(2\sqrt{2}\,t).$$

The initial conditions to find c_1 and c_2 are:

$$\underbrace{y(0) = \frac{1}{10}\,m}_{\text{because it starts from height } 10cm = \frac{1}{10}m}, \quad \text{and} \quad \underbrace{y'(0) = 0}_{\text{because the mass is released from a standstill, i.e. it initially has no velocity.}}$$

So

$$\frac{1}{10} = c_1 \cos(0) + c_2 \sin(0)$$

$$\Rightarrow c_1 = \frac{1}{10}$$

and $y'(0) = 0$ gives $\left(y' = 2\sqrt{2}\,c_1(-\sin(2\sqrt{2}t)) + c_2 2\sqrt{2}\cos(2\sqrt{2}t)\right)$.

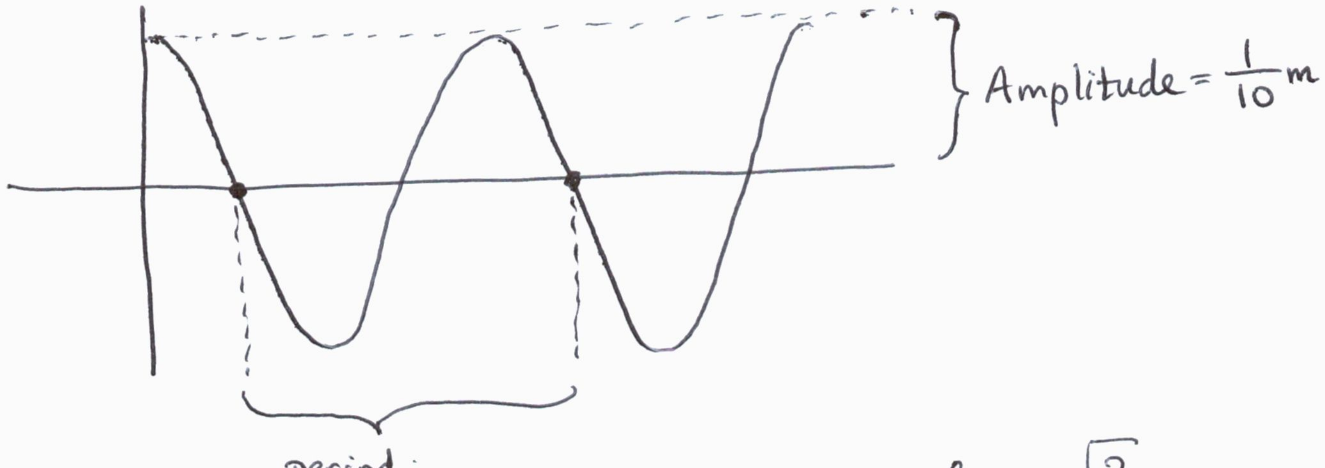
$$0 = -2\sqrt{2}\,c_1 \sin 0 + c_2 2\sqrt{2}\cos(0)$$

$$\Rightarrow 0 = c_2$$

So the motion is described by

$$y(x) = \frac{1}{10}\cos(2\sqrt{2}\,x).$$

which is:



$= 2\pi/2\sqrt{2} = \frac{\pi}{\sqrt{2}}$, and freq $= \frac{\sqrt{2}}{\pi}$

What if the object on the end of the spring is moving through a thick medium (such as water, instead of air) or worse, molasses?

Then it experiences a frictional force proportional to its speed:

$$F_{drag} = -\beta \frac{dx}{dt}, \quad \beta \text{ some positive constant which depends on the object and the medium.}$$

Then our application of Newton's Law

$$F = ma$$

gives, after regrouping:

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

<u>When we account for drag, what happens to the solutions?</u>

<u>Example:</u> Same problem as before, but with a drag of $\beta = 8$. Then describe the path of the object:

<u>Solution:</u> The equation of motion is

$$2\frac{dx^2}{dt^2} + 8\frac{dx}{dt} + 16x = 0$$

$$\Rightarrow \frac{dx^2}{dt^2} + 4\frac{dx}{dt} + 8x = 0$$

So the complementary equation is

$$m^2 + 4m + 8 = 0$$

$$\Rightarrow \text{roots are} \quad r_1, r_2 = \frac{-4 \pm \sqrt{4^2 - 8(4)(1)}}{2}$$

$$= \frac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i$$

So the general solution is

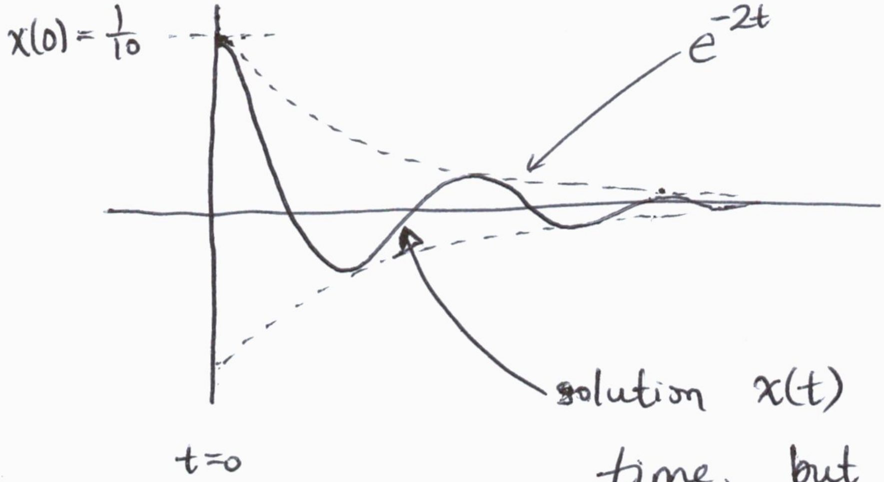
$$\cancel{y(x) = e^{-2t}(c_1\cos(2t) + c_2\sin(2t))}$$

$$x(t) = e^{-2t}(c_1\cos(2t) + c_2\sin(2t))$$

Then imposing the same initial conditions of $x(0) = \frac{1}{10}$, $x'(0) = 0$ gives

$$x(t) = \frac{1}{10} e^{-2t} \left(\cos(2t) + \sin(2t) \right).$$

Then the plot is:

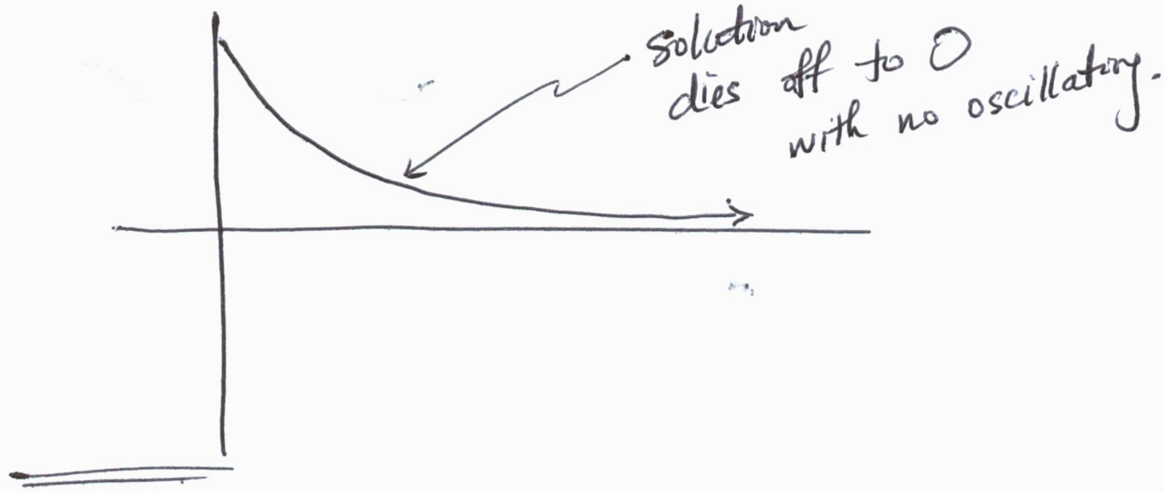


solution $x(t)$ dies off to zero over time, but oscillates all the way...

If the drag is <u>really</u> big, ie. $b \gg 0$, then $b^2 - 4ac$ will be positive and the complementary equation will have 2 real roots. Then the solution will be exponentials only, like:

$$x(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}.$$

Then the plot of the solution over time will be something like:



Summary: - With no drag, we get sines/cosines and it oscillates forever.

- With small drag we get e^{-at} (sines + cosines) and the solution dies off with some oscillations

- With large drag we get only exponentials, and the solutions die off with no oscillating.

§15.10 Springs and LCR examples continued.

Last day, we derived

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

as an equation describing the motion of a mass on a spring. We ended by observing that if β is big (ie there's a big drag on the object) then

$$Mm^2 + \beta m + k = 0$$

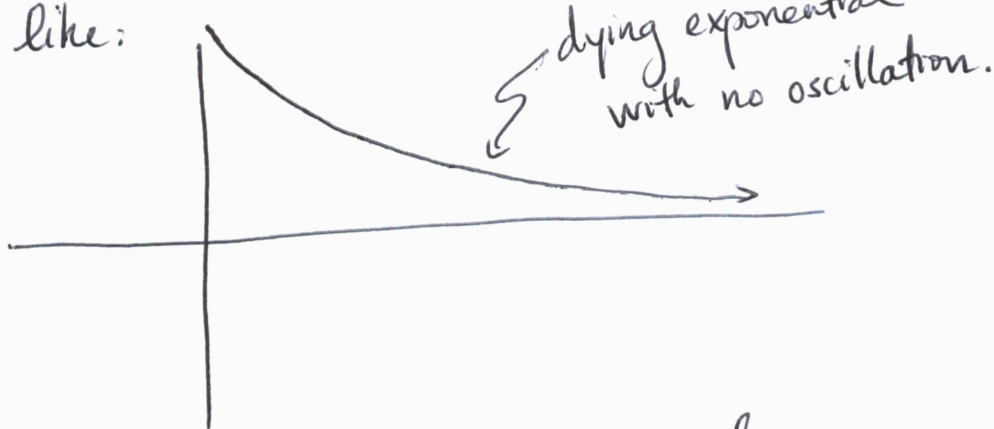
or

$$m^2 + \frac{\beta}{M}m + \frac{k}{M} = 0$$

will have only real roots, so the solution is

$$y(x) = c_1 e^{-r_1 x} + c_2 e^{-r_2 x}$$

where r_1 and r_2 are real roots. So the solutions die off like:



We could also introduce "forcing" of the mass:

Then if the Forcing $F(t)$ is a force independent of $x(t)$, we get

$$M\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=F(t)$$

<u>Example</u>: Suppose that $M=2$, $\beta=8$ and $k=16$ so that we have the same problem as last day. Moreover there is a machine attached to M which supplies 2N of upward force on the mass as it moves. Calculate $x(t)$.

<u>Solution</u>: From $F(t)=2$ and the values listed, we get

$$2\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=2$$

So the homogeneous problem $2\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=0$ has solutions of the form

$$x_h(t)=e^{-2t}(c_1\cos(2t)+c_2\sin(2t)).$$

Then the right hand side leads us to guess a constant as the particular solution:

$x_p(t)=A$. Then $x_p'(t)=x_p''(t)=0$ so substituting

$$2\cdot 0+8\cdot 0+16(A)=2$$

$$\Rightarrow A=\frac{1}{8}.$$

So $x_p(t) = \frac{1}{8}$ and our general solution is

$$x(t) = e^{-2t}\left(c_1\cos(2t) + c_2\sin(2t)\right) + \frac{1}{8}.$$

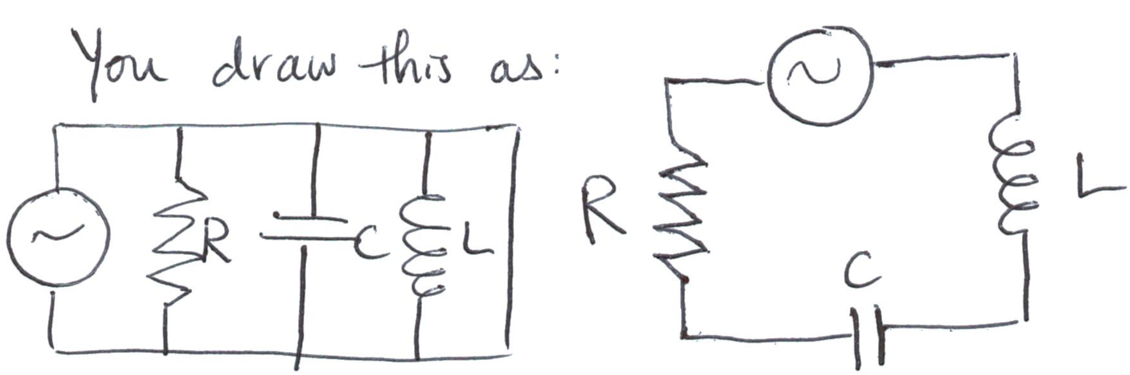
In other words, we find the same behaviour as before (in terms of oscillations and exponential decay) but the movement is shifted up by 1/8 by the upwards force of 2N.

Circuits example (RLC or LCR circuits).

Suppose you have an electrical circuit consisting of three things in series:

- An inductor of inductance L (measured in units H) Henry
  or ohms·seconds
- A capacitor of capacitance C (measured in F, Farads)
  or $\frac{\text{seconds}}{\text{ohms}}$
- A resistor of resistance R (measured in Ohms)

You draw this as:



(circuit diagrams)

The circle indicates a voltage source, and we'll write $I(t)$ for the current flowing around the circuit at any given time.

Analyzing the drop in voltage across each of the resistor, capacitor and inductor, we find that:

Across resistor: IR

Across capacitor: Q/C where $Q = \int I(t)$

Across inductor: $L\frac{dI}{dt}$.

These are physical laws that arise from studying the behaviour of electricity.

Then Kirchoff's law says that these quantities must sum to zero:

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int I(t) = 0.$$

But then we differentiate and get:

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0 \quad \text{(after dividing by } L\text{)}.$$

as a description of the current through the circuit at time t.

You'll see this written as

$$\frac{d^2I}{dt^2} + \nu\frac{dI}{dt} + \omega_0^2 I = 0$$

or $\ddot{I} + \nu\dot{I} + \omega_0^2 I = 0$ (Physicists use dots for derivatives wrt time).

Now for a ~~pendulum,~~ it is difficult to imagine a machine applying a constant 2N of force or maybe applying an oscillating force to the mass (when would this ever come up?).

However, for LCR circuits this is very natural:

In general we have DC (current) direct
and AC alternating.

And, for example, alternating current hooked up to an LCR circuit will naturally provide a sinusoidal forcing term in the DE above.

Example: (From text).

Suppose we have an LCR circuit with

$R = 25\,\Omega$.

$C = 0.01\,F$

$L = 2\,H$.

(powered)
and it is forced by an alternating current described by $10\sin(5t)$, which is initially all unpowered.
Model the system's current over time.

Solution:

The DE is:

$$2\frac{d^2I}{dt^2} + 25\frac{dI}{dt} + 100I = 10\sin(5t).$$

and everything is initially unpowered, so $\left.\frac{dI}{dt}\right|_0 = 0$,
$I(0) = 0$.

Then we get an auxiliarly eqn

$$2m^2 + 25m + 100 = 0$$

$$\Rightarrow m = \frac{-25 \pm \sqrt{625-800}}{4} = \frac{-25 \pm 5i\sqrt{7}}{4}.$$

and a homogeneous solution

$$I_h(t) = e^{-\frac{25}{4}t}\left(c_1\cos\left(\frac{5\sqrt{7}}{4}t\right) + c_2\sin\left(\frac{5\sqrt{7}}{4}t\right)\right).$$

We use undetermined coefficients and guess

$$I_p(t) = A\sin(5t) + B\cos(5t)$$

and plug $I_p'(t)$, $I_p''(t)$ into the DE and get

$$I_p(t) = \frac{2}{145}\left[2\sin(5t) - 5\cos(5t)\right].$$

So the current of the circuit is

$$I(t) = I_h(t) + I_p(t),$$

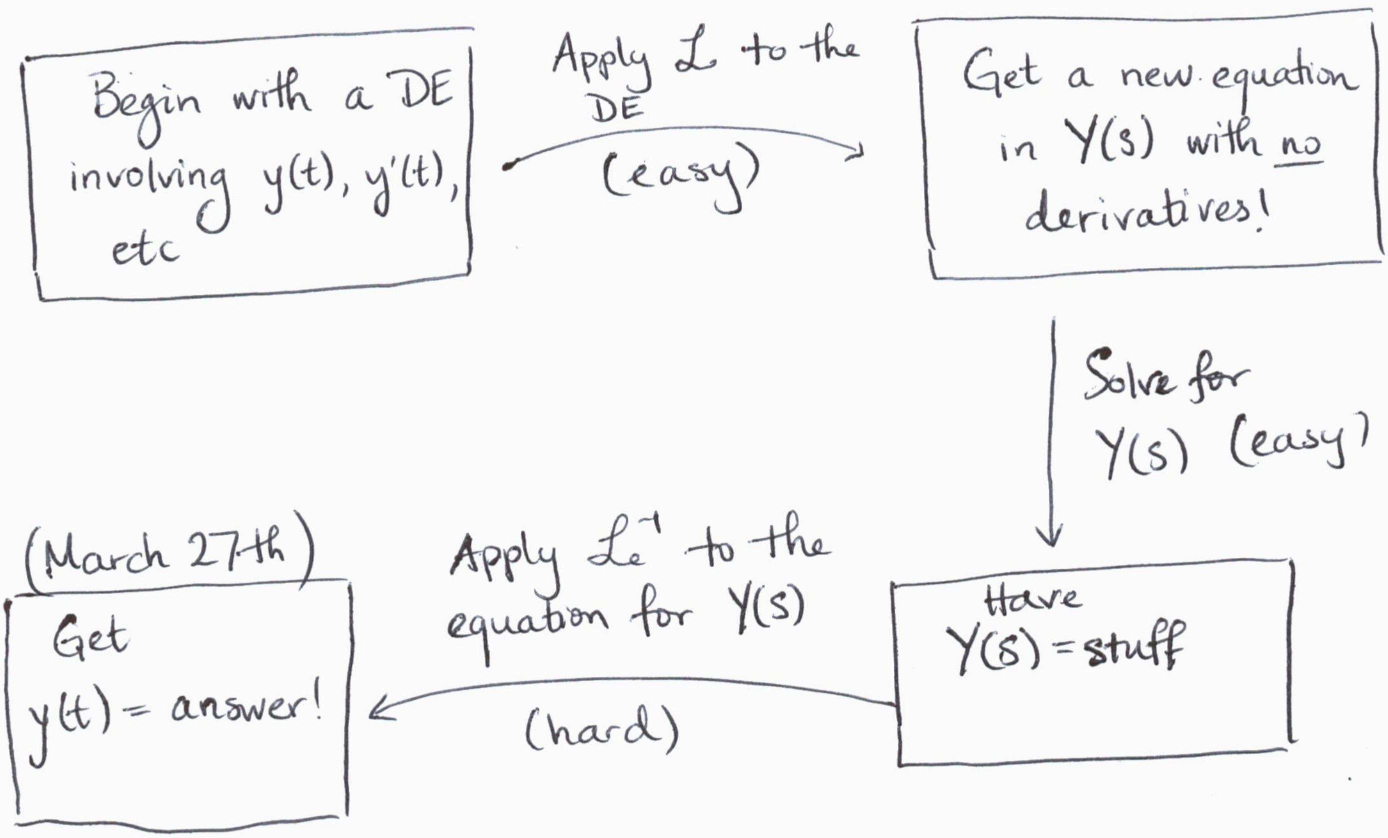
a superposition of two solutions.

Note that the homogeneous part "dies off" over time and the I_p part dominates as $t \to \infty$.

§16.1 MATH 2132 Friday, March 13

Laplace transforms, definitions and examples.

The Laplace transform is a linear operator that we denote as "$\mathcal{L}$". When you plug a function $y(t)$ into $\mathcal{L}$, it changes $y(t)$ into a new function $Y(s)$. Once we define the operator $\mathcal{L}$, here is how we will use it:



So first, we need to learn what $\mathcal{L}$ is and how to "plug functions into" $\mathcal{L}$.

First, when we write $\int_a^\infty f(t)\,dt$, what we mean is

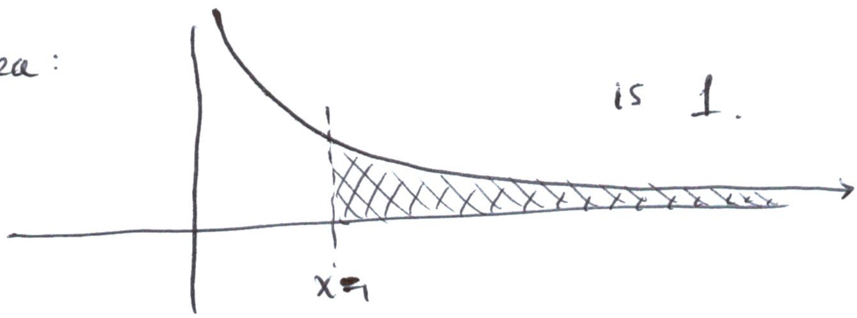
$$\lim_{b\to\infty} \int_a^b f(t)\,dt.$$

Example: What is $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$?

Solution: This integral is a limit:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \cancel{\frac{-1}{b}}^{\,0} - \left(\frac{-1}{1}\right) = 1.$$

So this area:  is 1.

Now the Laplace transform of a function $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt,$$ provided that the integral on the right exists.

Example: What is $\mathcal{L}\{e^{at}\}$, $a \in \mathbb{R}$?

Solution:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{at-st}\, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt$$

$$= \lim_{b\to\infty} \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^b$$

$$= \lim_{b\to\infty} \frac{1}{a-s}\left(e^{(a-s)b} - 1\right)$$

goes to zero if $a-s<0 \Rightarrow a<s$, otherewise DNE

$$= \frac{1}{a-s}(0-1) = \frac{1}{s-a}, \quad s>a.$$

Example: Find the Laplace transform of $f(t)=t$.

Solution: The definition is

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\cdot t\, dt = \lim_{b\to\infty} \int_0^b t\, e^{-st}\, dt$$

$$= \lim_{b\to\infty} e^{-st}\left(\frac{-1}{s^2} - \frac{t}{s}\right)\Bigg|_0^b$$

$$= \lim_{b\to\infty} \cancel{e^{-sb}\left(\frac{1}{s^2} - \frac{b}{s}\right)}^{\to 0} - e^0\left(\frac{-1}{s^2} - \frac{0}{s}\right) = \frac{1}{s^2}$$

if $s>0$.

The useful thing about Laplace transforms is that our procedure for solving DE's will work even when some of the functions involved are discontinuous or otherwise "badly behaved".

<u>Example</u>: Calculate the Laplace transform of

$$f(t) = \begin{cases} 2t^2 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1 \end{cases}$$

<u>Solution</u>: The Laplace transform is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

must break this into two integrals since $f(t)$ is defined in two different ways on different parts of the real line.

$$= \int_0^1 e^{-st} 2t^2\, dt + \int_1^\infty e^{-st} \cdot 1\, dt$$

$$= \left[e^{-st} \left(\frac{-4}{s^3} - \frac{4t}{s^2} - \frac{2t^2}{s} \right) \right]_0^1 + \lim_{b\to\infty} \int_1^b e^{-st}\, dt$$

$$= \underbrace{-e^{-s}\left(\frac{2}{s} + \frac{4}{s^2} + \frac{4}{s^3}\right)}_{\text{from } t=1} + \underbrace{\frac{4}{s^3}}_{\text{from } t=0} + \lim_{b\to\infty} \left[-\frac{e^{-st}}{s} \right]_1^b$$

$$= -e^{-s}\left(\frac{2}{s} + \frac{4}{s^2} + \frac{4}{s^3}\right) + \frac{4}{s^3} + \lim_{b\to\infty}\left(\cancel{\frac{-e^{-sb}}{s}}^{\to 0} + \frac{e^{-s}}{s}\right)$$

$$= -e^{-s}\left(\frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3}\right) + \frac{4}{s^3}, \text{ provided } s>0.$$

So, with a discontinuous function $f(t)$ as in the previous example, we have no way to solve things like:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

but with Laplace transforms this will be very easy.

Example: In general, for the Laplace transform to not exist, the function has to be somewhat crazy – specifically it has to grow faster than any exponential Me^{at}, M, a positive constants.

So, e.g. e^{t^2} grows faster than any function Me^{at}. To formally show that $\mathcal{L}\{e^{t^2}\}$ does not exist is quite hard, however since we cannot evaluate $\int_0^\infty e^{-st}e^{t^2}\,dt$ using elementary functions.

Note that since integrals satisfy:

$$\int f+g \, dt = \int f \, dt + \int g \, dt$$

and $\int cf \, dt = c \int f \, dt$

so do Laplace transforms, so we get

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

and $\mathcal{L}\{cf\} = c\,\mathcal{L}\{f\}$.