MATH 2132 §16.3 and 16.4.

Example: Use Laplace transforms to solve

$$y'' + y = f(t), \quad \text{where } y(0) = y'(0) = 0$$

and

$$f(t) = \begin{cases} \cos t & \text{if } 0 < t < \frac{3\pi}{2} \\ \sin t & \text{if } t > \frac{3\pi}{2} \end{cases}$$

Solution: We write $f(t)$ using step functions, and get

$$f(t) = \cos t\left(1 - h\left(t - \tfrac{3\pi}{2}\right)\right) + \sin t\left(h\left(t - \tfrac{3\pi}{2}\right)\right)$$

Then

Step 1: Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\cos t\} - \mathcal{L}\{h(t - \tfrac{3\pi}{2})\cos t\}$$
$$+ \mathcal{L}\{h(t - \tfrac{3\pi}{2})\sin t\}$$

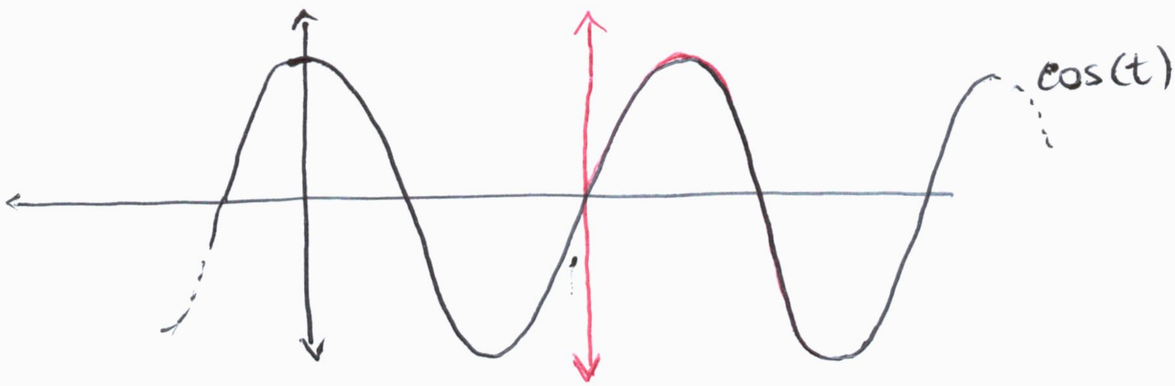
$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{s}{s^2+1} - e^{-\frac{3\pi}{2}s}\mathcal{L}\{\cos(t + \tfrac{3\pi}{2})\}$$
$$+ e^{-\frac{3\pi}{2}s}\mathcal{L}\{\sin(t + \tfrac{3\pi}{2})\}$$

Now, to take $\mathcal{L}$ of the shifted sines/cosines we need to simplify them:

$$\cos(t + \tfrac{3\pi}{2}) = \sin(t), \quad \sin(t + \tfrac{3\pi}{2}) = -\cos(t)$$

These follow from either

1. The angle sum formulas, or
2. The graphs. E.g. graph $\cos(t+\frac{3\pi}{2})$.



This is where the y-axis would be if cos(t) were shifted to give $\cos(t+\frac{3\pi}{2})$. In this way we get the graph of sin t

So now Laplace transforms give:

$$(s^2+1)Y(s) = \frac{s}{s^2+1} - e^{-\frac{3\pi}{2}s}\frac{1}{s^2+1} - e^{-3\pi/2}\frac{s}{s^2+1}$$

and then <u>Step 2</u> Solve for $Y(s)$:

$$Y(s) = \frac{s}{(s^2+1)^2} - e^{-\frac{3\pi}{2}s}\left(\frac{1}{(s^2+1)^2} + \frac{s}{(s^2+1)^2}\right).$$

So now

<u>Step 3</u>: Calculate $\mathcal{L}^{-1}$ of $Y(s)$.

Here, we're <u>actually stuck</u>!

When you do partial fractions, you get four ~~three~~ kinds of terms:

① $\frac{1}{s-a}$ $\rightsquigarrow$ inverse Laplace gives exponentials

② $\frac{1}{(s-a)^k}$ $\rightsquigarrow$ inverse Laplace gives exponentials multiplied by powers of t.

③ $\frac{1}{as^2+bs+c}$, where as^2+bs+c has no real roots

Then inverse Laplace gives sines/cosines by completing the square.

④ "Rarely", you also get

$\frac{1}{(as^2+bs+c)^k}$, where as^2+bs+c has no real roots.

There is nothing we have learned so far that deals with this. Moreover, as our last example showed, this situation <u>can</u> arise naturally.

<u>Convolution</u>: From Merriam-Webster:

"something that is very complicated and difficult to understand" (that about sums it up).

Convolution is a way of calculating the inverse Laplace transform of a product of two functions, like

$$\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$$

Suppose that $g(t)$ has Laplace transform $G(s)$ and $f(t)$ has Laplace transform $F(s)$. Then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)\,du.$$

The right hand side above is called "the convolution of f and g" and the integral is usually denoted $f*g$:

$$f*g = \int_0^t f(u)g(t-u)\,du$$

Example: Calculate the convolution of sin and cos.

Solution: The integral is

$$\cos(t) * \sin(t) = \int_0^t \cos(u)\sin(t-u)\,du$$

use $\sin(t-u) = \sin t \cos u - \sin u \cos t$

so $$= \int_0^t \cos u(\sin t \cos u - \sin u \cos t)\,du$$

So

$$f*g = \int_0^t \sin t \cos^2 u - \cos t \sin u \cos u \, du$$

$$= \int_0^t \sin t \cos^2 u \, du - \int_0^t \cos t \sin u \cos u \, du$$

$$= \sin t \int_0^t \cos^2 u \, du - \cos t \int_0^t \sin u \cos u \, du$$

$$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

set $v = \sin u$

$\Rightarrow dv = \cos u \, du$

$$= \frac{\sin t}{2} \int_0^t 1 + \cos 2u \, du - \cos t \int_0^{\sin t} v \, dv.$$

$=$ some work and some trig identities

$$= \frac{1}{2} t \sin t.$$

So, this means that if

$F(s) = \dfrac{s}{s^2+1}$ is the Laplace transform of cosine and

$G(s) = \dfrac{1}{s^2+1}$ is the Laplace transform of sine

Then $\mathcal{L}^{-1}\{F(s)\,G(s)\} = f * g$ so

$$= \frac{1}{2}\, t \sin t.$$

So, if we return to our original problem we are stuck at

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - \mathcal{L}^{-1}\left\{e^{-\frac{3\pi}{2}s}\,\frac{1}{(s^2+1)^2}\right\}$$

$$- \mathcal{L}^{-1}\left\{e^{-\frac{3\pi}{2}s}\,\frac{s}{(s^2+1)^2}\right\}$$

with what we have so far, this becomes

$$y(t) = \frac{1}{2}\, t \sin t \;-\; \mathcal{L}^{-1}\left\{e^{-\frac{3\pi}{2}s}\,\frac{1}{(s^2+1)^2}\right\}$$

$$- h\left(t - \frac{3\pi}{2}\right)\left(\frac{1}{2}\left(t - \frac{3\pi}{2}\right)\underbrace{\sin\left(t - \frac{3\pi}{2}\right)}_{\cos(t)}\right)$$

So all that's missing to finish the problem is

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$$

$$= \sin t * \sin t$$

$$= \frac{1}{2}(\sin t - t\cos t)$$

And then we get

$$y(t) = \frac{1}{2} t \sin t - h\left(t - \frac{3\pi}{2}\right)\left(\frac{1}{2} \overbrace{\sin\left(t - \frac{3\pi}{2}\right)}^{\cos(t)} - \left(t - \frac{3\pi}{2}\right)\overbrace{\cos\left(t - \frac{3\pi}{2}\right)}\right)$$

$$- h\left(t - \frac{3\pi}{2}\right)\left(\frac{1}{2}\left(t - \frac{3\pi}{2}\right)\cos(t)\right)$$

Convolution is not always so bad, and in fact can be downright easy in some cases.

Example: Compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$.

Solution: We can do this by partial fractions, or we can do:

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 1 * \sin t$$

$$= \int_0^t 1 \cdot \sin u \, du$$

$$= \Big[-\cos u\Big]_0^t$$

$$= -\cos(t) - (-\cos(0))$$

$$= 1 - \cos(t).$$

§16.3-4 Convolution continued.

Last day and yesterday in the tutorial, we saw that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$

where the convolution formula is

$$f*g = \int_0^t f(u)g(t-u)\,du.$$

Something that I've said but not formally checked yet is that "$*$" behaves like multiplication. Formally, the following things are true:

(i) $f*g = g*f$

(ii) $(kf)*g = f*(kg) = k(f*g)$, k any constant

(iii) $(f*g)*h = f*(g*h)$

(iv) $f*(g+h) = f*g+f*h$.

Each of these formulas, technically, should be checked. Here's how that would go: let's check (i), for example:

$$f*g = \int_0^t f(u)g(t-u)\,du$$

then set $v = t-u$, so $dv = -du$.

when $u=0$, then $v=t-0=t$

and when $u=t$, $v=t-t=0$.

So our integral becomes

$$\int_t^0 f(v-t)g(v)(-dv)$$

$$= \int_0^t f(v-t)g(v)\,dv = g*f$$

swithc order
if you want it to
look like the formula.

Example: Solve

$$y''-4y'+5y=2e^t+4, \quad y(0)=0, \quad y'(0)=0$$

using Laplace transforms and convolution.

Solution: Since we'll be using convolution, let's leave $\mathcal{L}\{2e^t+4\}$ and not evaluate it. That means we get:

Step 1: Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\}-4\mathcal{L}\{y'\}+5\mathcal{L}\{y\}=\mathcal{L}\{2e^t+4\}$$

$$s^2Y(s)-sy(0)-y'(0)-4(sY(s)-y(0))+5Y(s)=\mathcal{L}\{2e^t+4\}$$

$\Rightarrow (s^2-4s+5)Y(s)$ + bunch of zeroes $= \mathcal{L}\{2e^t+4\}$.

<u>Step 2</u> Solve for $Y(s)$.

$$Y(s) = \mathcal{L}\{2e^t+4\} \cdot \frac{1}{s^2-4s+5}.$$

<u>Step 3</u> Calculate $\mathcal{L}^{-1}$ of $Y(s)$.

Since it's a quadratic on the bottom, we check if it factors:

$$b^2-4ac = (-4)^2-4(1)(5) = 16-20 = -4,$$

so no, it doesn't factor. So we complete the square:

$$\frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2+1}$$ and then the table entries with $(s-2)^2+1$ as denominator are... wait, one of them is $\frac{1}{(s-2)^2+1}$!

We calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = e^{2t}\sin t$. So

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{2e^t+4\} \cdot \frac{1}{s^2-4s+5}\right\}$$

$$= \mathcal{L}^{-1}\{\mathcal{L}\{2e^t+4\}\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+5}\right\}$$

$$= (2e^t+4) * e^{2t}\sin t.$$

$$= 2e^{t} * e^{2t}\sin t + 4 * e^{2t}\sin t$$

$$= 2(e^{t} * e^{2t}\sin t) + 4(1 * e^{2t}\sin t)$$

$$= 2\int_0^t e^{2u}\sin u \cdot e^{t-u}\,du + 4\int_0^t e^{2u}\sin u\,du$$

$$= 2e^{t}\int_0^t e^{u}\sin u\,du + 4\int_0^t e^{2u}\sin u\,du.$$

Each of these can be evaluated using the double integration by parts trick that I showed before.

$$= 2e^{t}\left[\frac{1}{2}e^{u}(\sin(u)-\cos(u))\right]_0^t + 4\left[\frac{1}{5}e^{2u}(2\sin u-\cos u)\right]_0^t$$

$$= 2e^{t}\left[\frac{1}{2}e^{t}(\sin(t)-\cos(t)) + \frac{1}{2}\right] + \frac{4}{5}\left[e^{2t}(2\sin t-\cos t)+1\right]$$

$$= \frac{13}{5}e^{2t}\sin t - \frac{9}{5}e^{2t}\cos t + e^{t} + \frac{4}{5}$$

If you're really excited about this: go back to the start of the previous example and solve it the "traditional" way, check that you get the same thing.

Example: Find the general solution to

$$y'' + 3y' + 2y = e^{t} f(t),$$

where $f(t)$ is a continuous function. Express your answer as an integral.

Solution: We proceed as before:

Step 1: Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{t} f(t)\}$$

$$s^2 Y(s) - s y(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \mathcal{L}\{e^{t} f(t)\}.$$

Since we are solving for a general solution we replace $y(0)$ with C_1, $y'(0)$ with C_2:

$$(s^2+3s+2)Y(s) - C_1 s - C_2 - 3C_1 = \mathcal{L}\{e^{t} f(t)\}.$$

Step 2: Solve for $Y(s)$.

$$Y(s) = \mathcal{L}\{e^{t} f(t)\} \cdot \frac{1}{s^2+3s+2} + \frac{C_1 s + C_2 + 3C_1}{s^2+3s+2}.$$

Step 3: Take $\mathcal{L}^{-1}$.

Here we need to know

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\Rightarrow 1 = As + 2A + Bs + B$$

$\Rightarrow A+B=0$

$2A+B=1 \Rightarrow 2A-A=1, \quad A=1, \ B=1.$

So
$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2} \quad \text{and}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = e^{-t} - e^{-2t}$$

and therefore

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{e^t f(t)\} \cdot \frac{1}{s^2+3s+2}\right\}$$

$$= e^t f(t) * (e^{-t} - e^{-2t}) = \int_0^t e^u f(u)\left(e^{-(t-u)} - e^{-2(t-u)}\right) du.$$

Then to finish we need

$$\mathcal{L}^{-1}\left\{\left(c_1(s+3)+c_2\right)\left(\frac{1}{s^2+3s+2}\right)\right\}.$$

As with many exercises in the book, this is unexpectedly very difficult. If you use

$$\left(c_1(s+3)+c_2\right)\left(\frac{1}{s^2+3s+2}\right)$$

$$= c_1\left(\frac{2}{s+1} - \frac{1}{s+2}\right) + c_2\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$$

Then it's possible to take $\mathcal{L}^{-1}$ and get

$$2c_1 e^{-t} - c_1 e^{-2t} + c_2 e^{-t} - c_2 e^{-2t}.$$

$$= (2c_1 + c_2)e^{-t} - (c_1 + c_2)e^{-2t}$$

which allows us to "replace constants" and get the general solution

$$y(t) = \underbrace{\int_0^t e^{u} f(u)\left(e^{-(t-u)} - e^{-2(t-u)}\right) du}_{y_p} + \underbrace{D_1 e^{-t} + D_2 e^{-2t}}_{y_h}$$