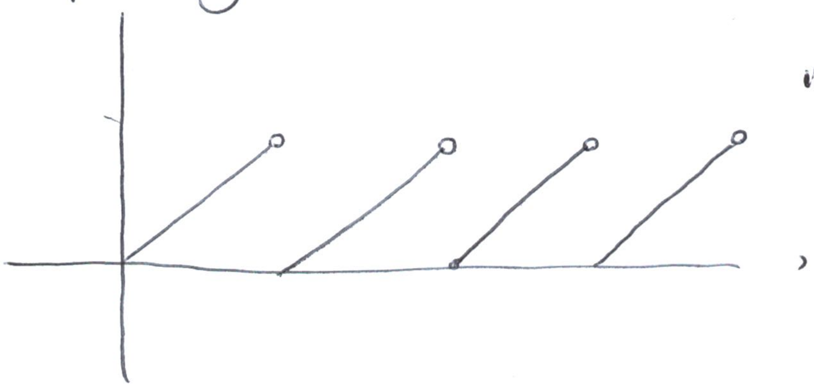
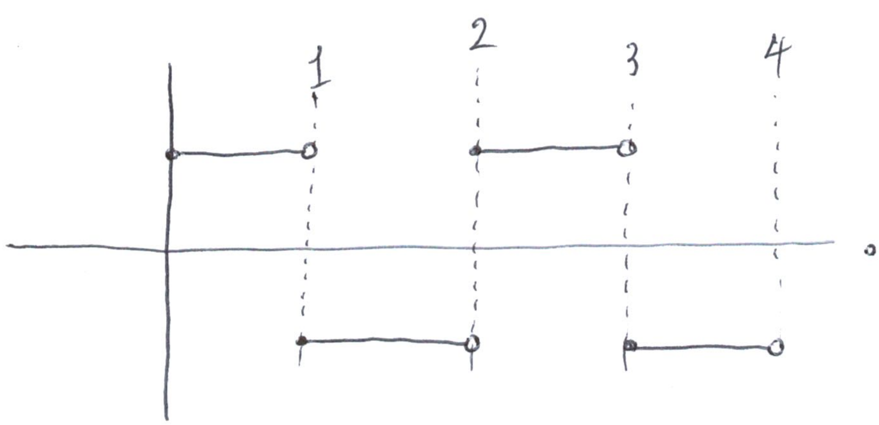
§16.2–16.3

Laplace transforms of periodic functions and derivatives.

A function which repeats, like sine or cosine, is called periodic. The length of 1 repeat is called the period, in the case of sine or cosine the period is 2π. We can have more general repeating functions though, like:



"sawtooth wave", or



"square wave". We can take the Laplace transform of functions like these, by writing each in terms of a single piece that gets infinitely repeated.

If the function $f(t)$ repeatse every p, then the Laplace transform is given by

$$F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt.$$

"Repeating every p" means $f(x+p) = f(x)\ \forall x$

<u>Example</u>: In the case of the square wave,

$$f(t) = \begin{cases} -1 & \text{if the greatest whole number less than } t \text{ is odd} \\ +1 & \text{if the greatest whole number less than } t \text{ is even} \end{cases}$$

and $f(t)$ repeats every 2, so the <u>period</u> $p=2$ in this case. So

$$\mathcal{L}e\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\, dt.$$

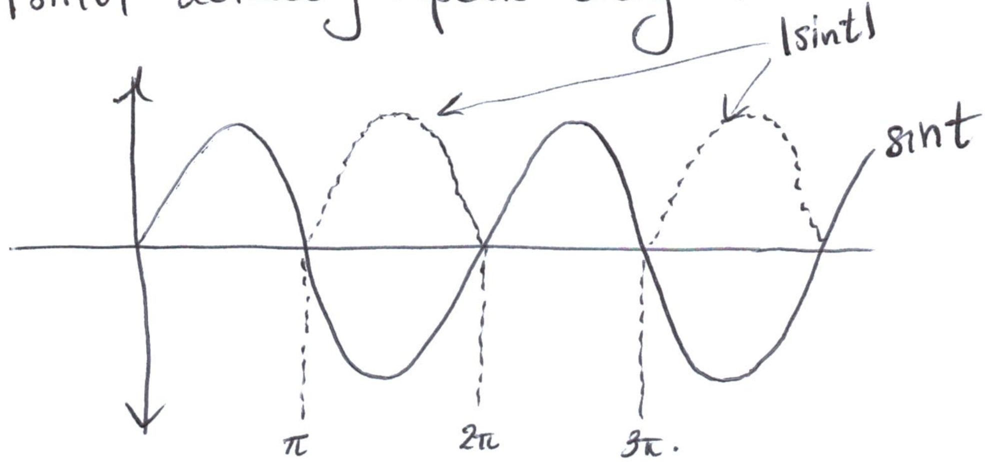
$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\, dt + \int_1^2 e^{-st}(-1)\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1-e^{-s}}{s} + \frac{e^{-2s}(1-e^{s})}{s} \right]$$

$$= \frac{1}{1-e^{-2s}} \cdot \left[\frac{1-2e^{-s}+e^{-2s}}{s} \right].$$

<u>Example</u>. Calculate the Laplace transform of $f(t) = |\sin t|$.

Solution: The function $\sin t$ repeats every 2π, but $|\sin t|$ actually repeats every π!



So we take $p = \pi$ and calculate:

$$\mathcal{L}\{|\sin t|\} = \left(\int_0^{\pi} e^{-st} |\sin t| \, dt\right) \cdot \frac{1}{1-e^{-\pi s}}$$

$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt \quad \text{(since } \sin t \text{ is positive there)}$$

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{e^{-\pi s} \mp 1}{s^2+1}\right]$$

$$= \frac{e^{-\pi s}+1}{(1-e^{-\pi s})(s^2+1)}.$$

If you continue on to do electrical applications of Laplace, these things can come up often.

Remark for the curious: The formula

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sp}} \int_0^{p} e^{-st} f(t) \, dt$$

is derived using geometric series, and its derivation on page 1133 is worth a look.

Now if you recall, the goal is to eventually use Laplace transforms to solve DE's, and the claim was that after applying Laplace to a DE, you get an equation with no derivatives in it.

So, how do you take $\mathcal{L}$ of a derivative and get an expression without derivatives?

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

now integrate by parts: $e^{-st}dt = dv$, $f(t) = u$

$$\Rightarrow v = \frac{-1}{s}e^{-st}, \quad \Rightarrow f'(t)dt = du$$

So by parts:

$$= \int_0^\infty e^{-st} f(t)\,dt = \left[-f(t)\cdot\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty \frac{-1}{s}e^{-st} f'(t)\,dt$$

$$= \lim_{b\to\infty}\left[-f(t)\cdot\frac{e^{-st}}{s}\right]_0^b + \frac{1}{s}\underbrace{\int_0^\infty e^{-st} f'(t)\,dt}_{\mathcal{L}\{f'(t)\}}$$

and what is $\lim_{b\to\infty} \left[-f(t) \frac{e^{-st}}{s} \right]_0^b$?

$$= \lim_{b\to\infty} \left[\underbrace{\cancel{-f(b)\frac{e^{-sb}}{s}}}_{\substack{\text{as long as} \\ \text{the Laplace transform} \\ \text{exists, this is zero}}} \to 0 + f(0)\frac{e^{-s\cdot 0}}{s} \right]$$

$$= \frac{f(0)}{s}.$$

So overall,

$$\mathcal{L}\{f(t)\} = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\{f'(t)\}.$$

Solving for $\mathcal{L}\{f'(t)\}$, we get

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

<u>Example</u>: Calculate the Laplace transform of the initial value problem:

$$2y' - 3y = \sin(3t), \quad y(0) = 1.$$

<u>Solution</u>: We're ready to take Laplace of the whole problem:

$$\mathcal{L}\{2y' - 3y\} = \mathcal{L}\{\sin(3t)\}.$$

$$\Rightarrow 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{\sin(3t)\}$$

$$\Rightarrow 2(sY(s) - y(0)) - 3Y(s) = \frac{3}{s^2+3^2}$$

(here, $Y(s) = \mathcal{L}\{y(t)\}$)

$$\Rightarrow 2sY(s) - 2 - 3Y(s) = \frac{3}{s^2+3^2}.$$

Note: The formula for taking Laplace of derivatives only applies if we have initial conditions at $t=0$!

Now, from the above we get:

$$2sY(s) - 3Y(s) = \frac{3}{s^2+3^2} + 2$$

$$\Rightarrow Y(s) = \left(\frac{3}{s^2+3^2} + 2\right)\frac{1}{2s-3}.$$

So a solution to this DE is some function $y(t)$ whose Laplace transform is as above. So, we'll need to take $\mathcal{L}^{-1}$ to find $y(t)$!

The derivative formula

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

is really only the beginning.

Before solving for $\mathcal{L}\{f'(t)\}$, our formula was:

$$\mathcal{L}\{f(t)\} = \frac{f(0)}{s} + \frac{1}{s} \underbrace{\int_0^\infty e^{-st} f'(t)\, dt}_{\mathcal{L}\{f'(t)\}},$$

which we got by integrating by parts. We can do integration by parts <u>again</u> and find $\mathcal{L}$ of higher derivatives, e.g.

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

and this pattern continues for higher derivatives.

§16.3 Laplace transforms and DE's.

We already saw from last day that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \qquad (*)$$

where $\mathcal{L}\{f(t)\} = F(s)$.

To take $\mathcal{L}$ of both sides of a DE, we also want to be able to take $\mathcal{L}$ of higher derivatives.

Here's how to do that: Since $f''(t)$ is the derivative of $f'(t)$, the formula $(*)$ above gives

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

then using $(*)$ again to replace $\mathcal{L}\{f'(t)\}$

$$\Rightarrow \mathcal{L}\{f''(t)\} = s(sF(s) - f(0)) - f'(0)$$

$$= s^2 F(s) - sf(0) - f'(0).$$

We can keep applying the same kinds of substitutions over and over to get a formula for the n^{th} derivative. It is:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

So now we can take the Laplace transform of <u>most</u> initial value problems.

Note that we still need the initial conditions to be given at zero:

$$f''(0) = ?, \; f'(0) = ??, \; f(0) = ???$$

Otherwise the formula doesn't apply.

<u>Example</u>: Solve the DE

$$y'' - 2y' + y = 2e^t, \quad y(0) = y'(0) = 0$$

using Laplace transforms

<u>Solution</u>:

<u>Step 1</u>: Take Laplace of both sides:

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 2\mathcal{L}\{e^t\}$$

$$\Rightarrow \left(s^2 Y(s) - s\,y(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + Y(s) = 2\frac{1}{s-1}.$$

$$\Rightarrow s^2 Y(s) - 2sY(s) + Y(s) = \frac{2}{s-1}.$$

<u>Step 2</u>: Solve for $Y(s)$:

$$Y(s)(s^2 - 2s + 1) = \frac{2}{s-1}$$

$$\Rightarrow Y(s) = \frac{2}{s-1} \cdot \frac{1}{s^2 - 2s + 1} = \frac{2}{(s-1)^3}$$

<u>Step 3</u>: Take $\mathcal{L}^{-1}$ to find $y(t)$:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}.$$

This is $\frac{1}{s^3}$, but shifted!

Our shift formula is:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \Rightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

So here, $F(s) = \frac{1}{s^3}$ and $a = 1$, so

$$y(t) = 2e^t\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 2e^t\left(\frac{t^2}{2}\right)$$ Look this up in tables.

$$= t^2e^t.$$

There is also a trick for finding <u>general solutions</u> to DEs using Laplace transforms.

It's this: If you are given a DE <u>without</u> initial conditions and asked to find the general solution, introduce $y(0) = C_1$, $y'(0) = C_2$, $y''(0) = C_3$, ... etc as initial conditions, and solve. The result is a general solution.

Example: Find the general solution to

$$y'' + y = h(t-2) - h(t-4)$$

Solution:

Right away we know the only way to do this is with Laplace transforms, because the RHS is discontinuous.

In order to find the general solution, we introduce the initial conditions $y(0) = C_1$, $y'(0) = C_2$.

Then:

Step 1: Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{h(t-2)\} - \mathcal{L}\{h(t-4)\}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}.$$

$$\Rightarrow s^2 Y(s) - C_1 s - C_2 + Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

Step 2: Solve for $Y(s)$:

$$(s^2+1)(Y(s)) = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s} + C_1 s + C_2$$

$$Y(s) = \frac{e^{-2s}}{s(s^2+1)} - \frac{e^{-4s}}{s(s^2+1)} + \frac{C_1 s}{s^2+1} + \frac{C_2}{s^2+1}.$$

Step 3: Take inverse Laplace.

We use $\mathcal{L}^{-1}\{e^{-as}F(s)\} = h(t-a)f(t-a)$ on the parts with exponentials appearing. To do this, we first need the inverse Laplace of $\dfrac{1}{s(s^2+1)}$

Partial fractions:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 1 = A(s^2+1) + Bs^2 + Cs$$

$$\Rightarrow \quad \begin{aligned} A+B&=0 \\ C&=0 \\ A&=1 \end{aligned} \qquad \Rightarrow B=-1.$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-s}{s^2+1}$$

So
$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\}$$
$$= h(t-2)(1-\cos(t-2))$$

similarly,

$$\mathcal{L}^{-1}\left\{e^{-4s}\frac{1}{s(s^2+1)}\right\} = h(t-4)(1-\cos(t-4)).$$

So overall, $\mathcal{L}^{-1}\{Y(s)\}$ gives

$$y(t) = h(t-2)(1-\cancel{\sin}\overset{\cos}{}(t-2)) - h(t-4)(1-\cos(t-4)) \Big\}\, y_p.$$

$$\cancel{+c_1 \sin t + c_2 \cos t} \quad + c_1 \cos t + c_2 \sin t \Big\}\, y_h$$

Remark: Note that even when we find a solution in this manner, the parts y_h and y_p are still clearly identifiable within the solution.

Example: Solve the initial value problem

$$y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^t,$$ with all initial conditions zero.

Solution:

The complementary equation factors as

$$m^4 - 4m^3 + 6m^2 - 4m + 1 = (m-1)^4,$$

so imagine if we had to do this using guess and check! Our guess would be $y_p(t) = t^5 e^t$, and then the derivatives... ugh. Here we do:

Taking Laplace, note

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s)$$ if all initial conditions are zero.

So after taking Laplace of both sides, we have

$$Y(s)(s^4-4s^3+6s^2-4s+1) = \frac{1}{s-1}$$

$$\Rightarrow Y(s)(s-1)^4 = \frac{1}{s-1}$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)^5}. \longleftarrow \frac{1}{s^5} \text{ shifted by } 1$$

Then taking inverse Laplace:

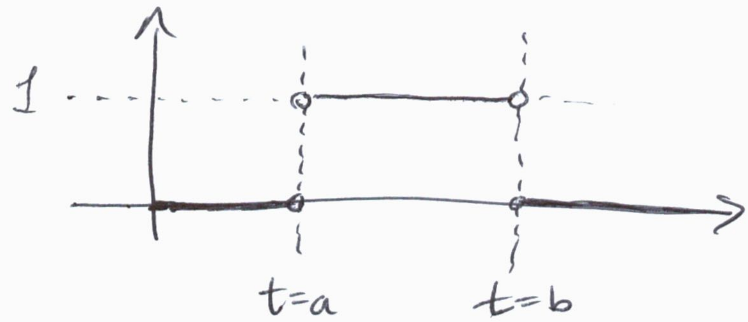
$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^5}\right\} = \frac{e^t t^4}{24}.$$

§ 16.3 Solving DE's using Laplace transforms, + discontinuos examples

We saw before that the function

$$h(t-a)-h(t-b)$$

is a function which is "on" between a and b:



We'll practice using this fact to compute $\mathcal{L}$e of piecewise-defined functions and solve the corresponding DE's.

~~Solut~~ Example: Find the general solution to

$$y''+4y=f(t), \text{ where } f(t)=\begin{cases} 0 & \text{if } 0\le t<5 \\ \frac{t-5}{5} & \text{if } 5<t<10 \\ 1 & \text{if } t>10. \end{cases}$$

Solution: We first write $f(t)$ as a sum of certain step functions to ~~take~~ its Laplace transform.

For each nonzero piece of $f(t)$, we use a function of the form $h(t-a)-h(t-b)$ to turn the piece "on" for the interval $[a,b]$. We get

$$f(t)=\big(h(t-5)-h(t-10)\big)\left(\frac{t-5}{5}\right)+1\cdot h(t-10).$$

Step 1

Nowewe take $\mathcal{L}$ of both sides:

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\mathcal{L}\{f(t)\}$$

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=\mathcal{L}\left\{h(t-5)\left(\frac{t-5}{5}\right)\right\}$$
$$-\mathcal{L}\left\{h(t-10)\left(\frac{t-5}{5}\right)\right\}$$
$$+\mathcal{L}\{h(t-10)\}.$$

Use the initial conditions $y(0)=c_1$, $y'(0)=c_2$ in order to make a general solution.

$$\Rightarrow (s^2+4)Y(s)=e^{-5s}\mathcal{L}\left\{\frac{t}{5}\right\}-e^{-10s}\mathcal{L}\left\{\frac{t+5}{5}\right\}+\mathcal{L}\{h(t-10)\}$$
$$+sC_1+C_2.$$

$$=\frac{e^{-5s}}{5s^2}-e^{-10s}\left(\frac{1}{5s^2}+\frac{1}{s}\right)+\frac{e^{-10s}}{s}$$
$$+C_1s+C_2$$

Step 2 Solve for $Y(s)$

$$\Rightarrow (s^2+4)Y(s) = \frac{e^{-5s}}{5s^2} - \frac{e^{-10s}}{5s^2} + C_1 s + C_2$$

$$\Rightarrow Y(s) = (e^{-5s} - e^{-10s})\frac{1}{5s^2(s^2+4)} + \frac{C_1 s}{s^2+4} + \frac{C_2}{s^2+4},$$

Step 3 Take $\mathcal{L}^{-1}$ of both sides.

We've got to use partial fractions on

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow 1 = As(s^2+4) + B(s^2+4) + Cs^3 + Ds^2$$

$$\Rightarrow A + C = 0 \Rightarrow C = 0.$$

$$B + D = 0 \Rightarrow D = -\tfrac{1}{4}$$

$$4A = 0 \Rightarrow A = 0$$

$$4B = 1 \Rightarrow B = \tfrac{1}{4}$$

So

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}s^{-2} - \frac{1}{4}\cdot\frac{1}{s^2+4}.$$

Now we do each piece individually:

(1) $\mathcal{L}^{-1}\left\{e^{-5s}\left(\frac{1}{5}\cdot\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)\right)\right\}$

$= \frac{1}{20}\mathcal{L}^{-1}\left\{e^{-5s}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)\right\}$

here, use $\frac{1}{2}\left(\frac{2}{s^2+4}\right)$

use:

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = h(t-a)f(t-a)$

$$= \frac{1}{20} h(t-5)\left[(t-5) - \frac{\sin(2(t-5))}{2}\right]$$

② $$\mathcal{L}^{-1}\left\{e^{-10s}\left(\frac{1}{5}\cdot\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)\right)\right\}$$

$$= \frac{1}{20} h(t-10)\left[(t-10) - \frac{1}{2}\sin(2(t-10))\right].$$

③ $$\mathcal{L}^{-1}\left\{\frac{C_1 s}{s^2+4}\right\} = C_1 \cos(2t)$$

④ $$\mathcal{L}^{-1}\left\{\frac{C_2}{s^2+4}\right\} = \frac{C_2}{2}\sin(2t) \rightsquigarrow \underbrace{C_2}_{\text{arbitrary constant.}} \sin(2t)$$

Overall,

$$y(t) = ① + ② + ③ + ④$$

$$\left.\begin{aligned} &= \frac{1}{20}\Big[h(t-5)\left[(t-5) - \tfrac{1}{2}\sin(2(t-5))\right] \\ &\quad - h(t-10)\left[(t-10) - \tfrac{1}{2}\sin(2(t-10))\right]\Big] \end{aligned}\right\} y_p.$$

$$\left.+ c_1\cos(2t) + c_2\sin(2t)\right\} y_h$$

<u>Example</u>. Solve

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1$$

if $f(t) = \begin{cases} 1 & \text{if } 0 < t < 3\pi \\ 0 & \text{if } t > 3\pi. \end{cases}$

<u>Solution</u>: The purpose of this example is to illustrate how important shifting properties of sine/cosine come up in the Laplace transform setting. Proceed as usual:

<u>Step 1</u>: Write $f(t) = 1 - h(t-3\pi)$. We can use 1 in place of the function "$h(t)$" since when we are doing Laplace transforms, we're only concerned with $t > 0$.

Then

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{h(t-3\pi)\}$$

$$\Rightarrow s^2 Y(s) - \overset{0}{\cancel{s\,y(0)}} - \overset{1}{y'(0)} + Y(s) = \frac{1}{s^{\cancel{2}}} - \frac{e^{-3\pi s}}{s}.$$

$$\Rightarrow (s^2+1)\,Y(s) = \frac{1}{s^{\cancel{2}}} - \frac{e^{-3\pi s}}{s} + 1.$$

<u>Step 2</u>: Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^{\cancel{2}}(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)} + \frac{1}{s^2+1}.$$

<u>Step 3</u> Take $\mathcal{L}^{-1}$. Since we have exponentials, we'll be using $\mathcal{L}^{-1}\{e^{-as}F(s)\} = h(t-a)f(t-a)$.

First,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

We write $f(t)$ using step functions, and use

$\mathcal{L}\{h(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$.

Write:

$$f(t) = \cos(t)(1-h(t-3\pi)) + \sin(t)(h(t-3\pi)-h(t-4\pi)) + \cos(t)(h(t-4\pi)).$$

Now, take Laplace of each piece:

$$\mathcal{L}\{\cos(t)-h(t-3\pi)\cos(t)\}$$

$$= \frac{s}{s^2+1} - e^{-3\pi s}\mathcal{L}\{\cos(t+3\pi)\}$$

$$= \frac{s}{s^2+1} - e^{-3\pi s}\mathcal{L}\{-\cos t\} = \frac{s}{s^2+1} + e^{-3\pi s}\cdot\frac{s}{s^2+1}$$

$$\mathcal{L}\{\sin t\, h(t-3\pi) - h(t-4\pi)\sin t\}$$

$$= \mathcal{L}\{h(t-3\pi)\sin t\} - \mathcal{L}\{h(t-4\pi)\sin t\}$$

$$= e^{-3\pi s}\mathcal{L}\{\sin(t+3\pi)\} - e^{-4\pi s}\mathcal{L}\{\sin(t+4\pi)\}$$

$$= e^{-3\pi s}\mathcal{L}\{-\sin t\} - e^{-4\pi s}\mathcal{L}\{\sin t\}$$

$$= -\frac{e^{-3\pi s}}{s^2+1} - e^{-4s\pi}\cdot\frac{1}{s^2+1}.$$

Last, $\mathcal{L}\{h(t-4\pi)\cos t\} = e^{-4\pi s}\mathcal{L}\{\cos t\}$

$$= e^{-4\pi s}\cdot\frac{s}{s^2+1}.$$

You also need to know things like:

$\sin(\theta+\frac{\pi}{2}) = \cos(x)$, $\cos(x+\frac{\pi}{2}) = \sin(x+\pi) = -\sin(x)$,

that sort of thing. Otherwise you cannot deal with shifted trig functions.

So $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$

and therefore

$$\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s(s^2+1)}\right\} = h(t-3\pi)\left[1 - \cos(t-3\pi)\right]$$

and last

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

Overall,

$$y(t) = t - \cos t - h(t-3\pi)\left[1 - \cos(t-3\pi)\right] + \sin t.$$

Now the shift $\cos(t-3\pi)$ actually gives $-\cos(t)$, since

$\cos(t-3\pi) = -\cos t$ so

$$\underline{y(t) = t - \cos t - h(t-3\pi)\left[1 + \cos t\right] + \sin t,}$$

Does this kind of identity matter? or is it purely cosmetic?

<u>Example</u>. Calculate the Laplace transform of

$$f(t) = \begin{cases} \cos t & \text{if } 0 < t < 3\pi \\ \sin t & \text{if } 3\pi < t < 4\pi \\ \cos t & \text{if } t > 4\pi. \end{cases}$$

<u>Solution</u>: